Consistent Indices*

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May 20, 2023

Abstract

In many decision problems, agents base their actions on a simple objective index, a single number that summarizes the available information about objects of choice independently of their particular preferences. This paper proposes an axiomatic approach for deriving an index which is objective and, nevertheless, can serve as a guide for decision making for decision makers with different preferences. Applications in specific decision-making environments include indices of riskiness, informativeness, performance, and delay. Indices derived using my approach are monotonic in standard partial orders like stochastic dominance and Blackwell informativeness.

*I have benefited from comments by many friends and colleagues. I am especially grateful to Nageeb Ali, Sergiu Hart, Jerry Green, Eric Maskin, Amnon Schreiber, Roberto Serrano, and Tomasz Strzalecki. Kazuyuki Higashi provided excellent research assistance. Financial support from the Geneva Association is greatfully acknowledged.

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1 Introduction

In many decision-making environments, agents base their actions on a simple *objective index*, a single number that summarizes the available information about objects of choice and does not depend on agents' particular preferences. Agents might choose to do this because they are having difficulty attaining and interpreting information or because they have an overabundance of useful information. For example, the *Sharpe ratio*—the ratio between the expected net return and its standard deviation—is frequently used as a performance measure for portfolios (Sharpe, 1966; Welch, 2008).

This paper proposes an axiomatic approach to deriving an index that is *objective* but nevertheless suitable as a guide for decision-making for decision makers with different preferences. The approach is developed in a general decision-making environment where an agent chooses whether to accept or reject a transaction (a gamble, a cashflow, etc.). It requires that the index (e.g., of riskiness or of delay) satisfy two properties.

The first property, *local consistency*, roughly requires that agents can optimally decide on each "small" transaction based on a threshold value of the index without knowing any other details about the transaction. Theorem 2 in Section 4 shows that such a cutoff must be monotonic in "local preferences" (e.g., absolute risk aversion Pratt, 1964). Even though in applications transactions are complex and multidimensional, this property is satisfied by many indices. However, some of the indices that satisfy this desirable property also have normatively undesirable properties. For example, the Sharpe ratio has such a property: it is not monotonic with respect to first-order stochastic dominance (outside of the domain of normal distributions).[1]

The second property required of the index links the value of the index on large transactions to its value on small transactions. For an index to satisfy this property, an agent whose threshold for accepting small transactions is always lower (higher) than a particular value of the index must likewise accept (reject) large transactions whose value is lower (higher) than this value. Section 5 provides the formal statement, which is inspired by Samuelson (1963), as well as a discussion of the normative appeal of this property. It then presents Theorem 3, which establishes that the two properties characterize a unique index for the respective decision problem (up to continuous monotonic transformations).[2]

Throughout the paper, I illustrate the usefulness of the results in two distinct domains: the riskiness of additive gambles (as studied by Aumann and Serrano, 2008), where I derive the Aumann–Serrano index of riskiness, and the delay of investment cashflows, where I derive the inverse internal rate of return (IRR) index of delay. To the best of my knowledge, no other paper in this stream of the literature has considered indices for this domain.

The appendix includes many other applications. For portfolio allocation problems, I derive the generalized Sharpe ratio (Hodges, 1998; Hellman and Schreiber, 2018). For information transactions, I derive the normalized value of information (Cabrales et al., 2017). I also derive indices of riskiness

[1] In the context of risk, Heller and Schreiber (2020) identify a similar requirement as *"a necessary condition for a plausible risk index"* and they *"leave for future research the interesting question of how to choose among the various risk indices that satisfy this necessary condition."* This paper responds to this challenge.

[2] To be precise, two additional mild conditions are required as well.

for multiplicative gambles (Schreiber, 2013) and menus of gambles (similar to an index proposed by Michaeli, 2014). Because the results are proved in a general, abstract model, they can be applied in a variety of other domains of interest.

Relation to the literature. This paper contributes to the literature on objective indices for specific decision problems, which dates back at least to Fisher (1930). It is most closely related to the pioneering work of Aumann and Serrano (2008) and the large body of literature that followed. Aumann and Serrano (2008) use a decision-theoretic, axiomatic approach to derive an index of riskiness for additive gambles.[3] The key axiom in Aumann and Serrano (2008) is the "duality" axiom. They explain:

> `The concept is based on that of risk aversion: We think of riskiness as a kind of ‘‘dual’’ to risk aversion—specifically, as that aspect of a gamble to which a risk-averter is averse. So on the whole, we expect individuals who are less risk averse to take riskier gambles. As Machina and Rothschild (2008, 7:193) put it, ‘‘risk is what risk-averters hate.’’`

There is a large literature applying duality-type axioms to other decision-making environments. Schreiber (2013) develops an index of relative riskiness for multiplicative gambles. Cabrales et al. (2017) study the value of information for investors.[4] Hellman and Schreiber (2018) study a standard portfolio allocation problem. Michaeli (2014) studies sets of additive gambles, focusing on ambiguity-averse decision makers.

The first step of my analysis is introducing a unifying framework that nests the abovementioned applications as well as many others. Theorem 1 in Section 3 shows that three properties characterize a unique index for the respective application (up to continuous monotonic transformations). These properties, which include a "weak duality" requirement and mild continuity and monotonicity requirements, are typically implied by axioms that have been used for domain-specific duality-type characterizations.

Duality-type axioms have a displeasing feature: they rely on interpersonal comparisons. For example, Aumann and Serrano's duality axiom requires *"individuals who are less risk averse to take riskier gambles."* My approach is not subject to this critique. However, Theorem 3 shows that the two properties that I require the index to satisfy imply the "weak duality" requirement from Theorem 1.

The idea that small decisions can be made based exclusively on the index has appeared in Schreiber (2016), who studies a continuous-time setup and restricts attention to short-term investments. Like this paper, Schreiber (2016) shows that many indices of riskiness have this property. Heller and Schreiber (2020) generalize this results to other decision-making environments. Neither paper provides guidance on how to choose from these indices. The present paper addresses this challenge by linking the value of the index on small transactions to its value on large transactions. A large body of

[3] Foster and Hart (2009) present a different index of riskiness with an operational interpretation. Hart (2011) demonstrates that both indices arise from a comparison of acceptance and rejection of gambles (see also Schreiber, 2015), and Foster and Hart (2013) develop alternative axiomatizations for both indices. Homm and Pigorsch (2012b) provide an operational interpretation of the Aumann–Serrano index of riskiness.

[4] See also Cabrales et al. (2013) and Shorrer (2018).

literature dating back at least to Samuelson (1963) and Pratt (1964) studies the connections between decisions on small and large transactions.

Though the literature on indices focuses on complete orders, a large body of work focuses on partial orders over which all "reasonable" agents agree. Notable examples include first- and second-order stochastic dominance (Hanoch and Levy, 1969; Hadar and Russell, 1969; Rothschild and Stiglitz, 1970) in the context of gambles, Blackwell's (1953) order over information structures, stochastic dominance in the presence of a risk-free asset (Levy and Kroll, 1978) in the context of portfolio allocation, and time dominance (Bøhren and Hansen, 1980; Ekern, 1981) in the context of cashflows. In each of the applications I present, the index I derive is monotonic in the corresponding partial order. As I explain in Section 6, this is not a coincidence but rather a feature that future applications will also possess.

Apart from serving as input in decision-making, indices are used to limit the discretion of agents by regulars (Artzner, 1999) or those who delegate (Turvey, 1963). For example, a mutual fund manager may be required to invest in bonds that are rated AAA. Indices are also used in empirical and theoretical studies to summarize complex, multidimensional attributes (e.g., Echenique and Fryer, 2007). My approach can potentially be used to derive indices in some of these settings.

2 Framework

A *decision-making environment* consists of a set of *decision problems* in which a *decision maker* is offered a *transaction* that she can either accept or reject.

Transactions. A *transaction* is a pair $(\mu, \mathbf{x})$ where $\mu \in \mathbb{R}_{++}$ and $\mathbf{x}$ belongs to an abstract space X. In applications, μ is often interpreted as a payment that the agent makes (receives) in order to obtain $\mathbf{x}$. The set of all possible transactions, T, is a subset of $\mathbb{R}_{++} \times X$.

Decision makers. The set of *decision-maker* types, DM, is parameterized by a continuous function $C : \mathbb{R} \to \mathbb{R}_{++}$ and a status quo $w \in \mathbb{R}$. In applications, the function may describe a utility function while the status quo may capture the agent's current wealth level. I sometimes refer to $(f(\cdot), w)$ as a decision maker $f(\cdot)$ with status quo w. I assume that if $(f(\cdot), w) \in$ DM, then $(f(\cdot), w') \in$ DM for any status quo w'.

Fixed decision types. The set DM includes the decision makers labeled by $f(\cdot) \equiv c$ for all $c \in \mathbb{R}_{++}$ (at every status quo). The decisions of these agents do not depend on the status quo. Formally, for any transaction $(\mu, \mathbf{x}) \in T$, any $c > 0$, and any $w, w' \in \mathbb{R}$, the decision maker $f(\cdot) \equiv c$ with status quo w accepts $(\mu, \mathbf{x}) \in T$ if and only if the decision maker $f(\cdot) \equiv c$ with status quo w' accepts $(\mu, \mathbf{x})$. I refer to these agents as *fixed decision types*.

Monotonicity in types. For each status quo, decision makers' acceptance and rejection decisions are monotonic in their types (using the partial order that compares functions pointwise). Formally, for any $(\mu, \mathbf{x}) \in T$ and $w \in \mathbb{R}$, for any pair of decision makers $(f(\cdot), w)$ and $(h(\cdot), w)$, if $f(\cdot) \geqslant h(\cdot)$ and $(f(\cdot), w)$ accepts $(\mu, \mathbf{x})$, then so does $(h(\cdot), w)$.

Monotonicity in μ. For any $(\mu', \mathbf{x}) \in T$, for any decision maker, there exists $\mu^* \in \mathbb{R}_{++}$ such that the transaction $(\mu', \mathbf{x}) \in T$ is accepted if and only if $\mu' > \mu^*$.[5]

Richness of preferences. For any $(\mu, \mathbf{x}) \in T$, there exists $c^*(\mu, \mathbf{x}) \in \mathbb{R}_{++}$ such that $f(\cdot) \equiv c$ with status quo w rejects $(\mu, \mathbf{x})$ if and only if $c \geqslant c^*(\mu, \mathbf{x})$.[6] Furthermore, if $\mu' > \mu$, then $c^*(\mu', \mathbf{x}) > c^*(\mu, \mathbf{x})$.

Richness of transactions. For any $(\mu, \mathbf{x}) \in T$, there exists $\varepsilon > 0$ such that if $|\mu' - \mu| < \varepsilon$, then $(\mu', \mathbf{x}) \in T$. Furthermore, for any $c \in \mathbb{R}_{++}$, there exists μ' such that $(\mu', \mathbf{x}) \in T$ and $c^*(\mu', \mathbf{x}) = c$.

2.1 An Illustration via the Additive Gambles Setting

This section shows that the canonical setting of additive gambles (Aumann and Serrano, 2008; Foster and Hart, 2009, 2013; Hart, 2011) is a special case of the framework. Subsequent sections consider several other important settings.

Agents' preferences are summarized by a von Neumann–Morgenstern *utility function* for money and a *status quo wealth*. Utility functions are strictly increasing, strictly concave, and twice continuously differentiable. The transactions under consideration are gambles. A *gamble* $\mathbf{g}$ is a finite-valued real random variable with positive expectation and some negative values (i.e., $\mathbb{E}[\mathbf{g}] > 0$ and $\Pr\{\mathbf{g} < 0\} > 0$). The collection of all gambles is denoted by $\mathcal{G}$. A gamble $\mathbf{g} \in \mathcal{G}$ is *accepted* by u at wealth w if $\mathbb{E}[u(w + \mathbf{g})] > u(w)$ and is *rejected* otherwise.

For any gamble $\mathbf{g} \in \mathcal{G}$, $L(\mathbf{g})$ and $M(\mathbf{g})$, respectively, denote the maximal loss and gain from the gamble that occur with positive probability. Formally, $L(\mathbf{g}) := \max \operatorname{supp}(-\mathbf{g})$ and $M(\mathbf{g}) := \max \operatorname{supp}(\mathbf{g})$. Additionally, the *Arrow–Pratt coefficient of absolute risk aversion* (ARA), ρ, of u at wealth w is denoted by

$$\rho_u(w) := -\frac{u''(w)}{u'(w)}.$$

The additive gambles setting is a special case of the general model.

Transactions. Let X be the set of zero-mean, non-degenerate, finite-valued random variables and $T = \{(\mu, \mathbf{x}) \in \mathbb{R}_{++} \times X \mid \mu + \mathbf{x} \in \mathcal{G}\}$. Then the mapping taking $\mathbf{g}$ to $(\mathbb{E}[\mathbf{g}], \mathbf{g} - \mathbb{E}[\mathbf{g}])$ is a bijection between $\mathcal{G}$ and T.

Decision makers. The behavior of an agent with utility u and wealth w is fully pinned down by $\rho_u(\cdot)$ and status quo wealth w. Furthermore, $\rho_u(\cdot)$ is positive and continuous (because $u''(\cdot) < 0$ and $u'(\cdot) > 0$ and both are continuous).[7]

[5] Apart from monotonicity in μ, this requirement encodes a continuity or a tie-breaking assumption. Choosing a weak (rather than strict) inequality would yield the same results.

[6] To reduce notation, when there is no risk of confusion, I write $c^*(\mu, \mathbf{x})$ instead of $c^*((\mu, \mathbf{x}))$. Similarly, I write $Q(\mu, \mathbf{x})$ instead of $Q((\mu, \mathbf{x}))$ and so on.

[7] If $v = au + b$ for $a > 0$, then u and v make the same decisions and are mapped to the same type. Since Aumann and Serrano (2008) identify agents with utility functions, their model admits this multiplicity. This detail is inconsequential for my analyses (and theirs). I therefore ignore the question of whether there exist multiple copies of the same type here and in the rest of the applications.

Fixed decision types. Constant absolute risk aversion (CARA) agents (ones with $\rho_u \equiv c > 0$) are not subject to wealth effects—they make the same decisions at every wealth level.

Monotonicity in types. Pointwise higher types correspond to more concave utility functions. Monotonicity of decisions in types therefore follows from Jensen's inequality (see Pratt, 1964).

Monotonicity in μ. Monotonicity in μ follows from monotonicity of expected utility preferences with respect to first-order stochastic dominance. The strict inequality follows from the assumption that agents reject gambles when they are indifferent.

Richness of preferences. Richness of preferences follows from well-known properties of CARA functions. Specifically, the existence of a critical ARA level $c^*(\mu, \mathbf{g})$ under which a CARA agent is indifferent can be established by the intermediate value theorem, the uniqueness of this value by Jensen's inequality, and its monotonicity in μ by CARA functions being strictly increasing in wealth.

Richness of transactions. The first part holds since for any $\mathbf{g} \in \mathcal{G}$, for any $0 < \varepsilon < \min\{L(\mathbf{g}), \mathbb{E}[\mathbf{g}]\}$, we have $\mathbf{g} \pm \varepsilon \in \mathcal{G}$. The second part follows from the intermediate value theorem upon noting that for any level of ARA, $c > 0$, a CARA-c agent rejects gambles of the form $\mathbf{g} + x$ for x sufficiently close to $-\mathbb{E}(\mathbf{g})$ and accepts gambles of the form $\mathbf{g} + x$ for x sufficiently close to $L(\mathbf{g})$.

3 Characterizing Consistent Indices

I begin in Section 3.1 by stating three requirements for an index and showing that a single index meets these requirements for each decision-making environment that meets the conditions of my model. I apply this result to several decision-making environments. In Section 3.2, I derive the Aumann–Seranno index of riskiness for the setting where agents are offered additive gambles. In Section 3.3, I derive the inverse IRR index of delay for the classic capital budgeting problem in which agents are offered investment cashflows.

Appendix A includes additional applications that follow the same pattern. Appendix A.1 considers a standard portfolio allocation problem and derives the generalized Sharpe ratio (Hodges, 1998; Hellman and Schreiber, 2018), an index that coincides with the Sharpe ratio on the domain of normal distributions but differs from it outside this domain (as it is sensitive to higher-order moments). Appendix A.2 considers the setting of information acquisition by investors facing a standard investment problem (Arrow, 1972) and derives the normalized value of information (Cabrales et al., 2017). Appendix A.3 derives an index of riskiness for menus of gambles.[8] Appendix A.4 considers the setting of multiplicative gambles and derives Schreiber's (2013) index of relative risk aversion.

3.1 Characterization

An *index (for a decision-making environment)* is a function $Q : T \to \mathbb{R}_{++}$. I first define a basic monotonicity property for the index: the index $Q(\mu, \mathbf{x})$ should be decreasing in μ.

[8]The index I derive in Appendix A.3 generalizes Michaeli's (2014, Proposition 6) index for extremely optimistic agents. As I discuss, the focus of Michaeli (2014) is ambiguity-averse agents. My approach can be used to derive the indices he derives for ambiguity-averse agents as well as analogous indices for menus of multiplicative gambles.

Definition 1. *An index Q satisfies **Property M** if for every $(\mu, \mathbf{x})$ and $(\mu', \mathbf{x})$ in T, $\mu' < \mu$ implies that $Q(\mu, \mathbf{x}) < Q(\mu', \mathbf{x})$.*

Next, I define a basic continuity property for the index: for a fixed $\mathbf{x}$, the index $Q(\mu, \mathbf{x})$ should be continuous in μ.

Definition 2. *An index Q satisfies **Property C** if for every $(\mu, \mathbf{x})$ in T and a sequence of transactions $\{(\mu_n, \mathbf{x})\}_{n=1}^{\infty}$ in T, $\lim_{n\to\infty} \mu_n = \mu$ implies that $\lim_{n\to\infty} Q(\mu_n, \mathbf{x}) = Q(\mu, \mathbf{x})$.*

Finally, I introduce a weak duality property. Specifically, Property **WD** requires that the decisions of fixed decision types are monotonic in the index.

Definition 3. *An index Q satisfies **Property WD** if for any pair of fixed decision types $h(\cdot) \equiv c$ and $f(\cdot) \equiv c' \geqslant c$, any status quo w, and any pair of transactions $\mathbf{t}, \mathbf{t}' \in T$, if $h(\cdot)$ rejects $\mathbf{t}'$ at status quo w and $Q(\mathbf{t}) > Q(\mathbf{t}')$, then $f(\cdot)$ rejects $\mathbf{t}$ at status quo w.*

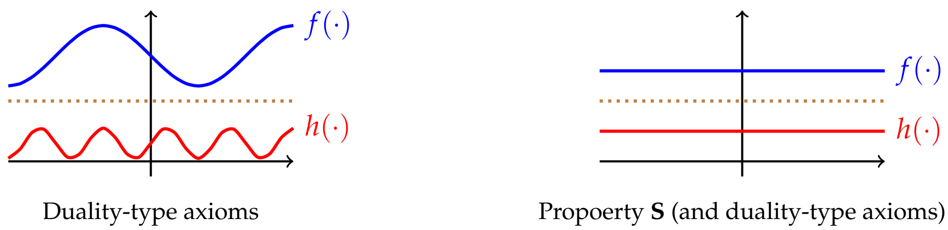

*Figure 1: Pairs of types whose decisions are required to be monotone in the index under Property **S** and under duality-type axioms.*

Property **WD** requires that if $f(\cdot)$ and $h(\cdot)$ are fixed decision types (as illustrated in the right panel of Figure 1), their decisions are monotonic in the index (higher types reject more transactions). This is a weaker requirement than that of duality-type axioms, which require monotonicity with respect to pairs of types that can be separated by a fixed decision type (as illustrated in the left panel of Figure 1). Hence, any justification for duality-type axioms is also a justification for Property **WD**.

Some readers may find Property **WD** (and duality-type axioms) displeasing because they rely on interpersonal comparisons. I revisit this issue in Section 5, replacing Property **WD** with other properties that such readers may find less objectionable.

Theorem 1. *The index Q satisfies Properties **M**, **C**, and **WD** if and only if $Q(\mathbf{t}) \equiv \phi(1/c^*(\mathbf{t}))$ for some continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$.*

Proof. ($\Longrightarrow$) Consider the index $Q(\cdot) := 1/c^*(\cdot)$. It is a well-defined index because $c^*(\cdot)$ is well defined and positive for any transaction. I show that it satisfies all three properties:

- <u>Q satisfies Property **M**</u>: Follows from richness of preferences.

- Q satisfies Property **C**: Assume that $\mu_n \to \mu'$ and $\mu_n > \mu'$ for each n. In this case, $c^*(\mu_n, \mathbf{x}) > c^*(\mu', \mathbf{x})$ for each n (by monotonicity of preferences in μ), and so $\limsup_{n\to\infty} c^*(\mu_n, \mathbf{x}) \geqslant c^*(\mu', \mathbf{x})$. Toward contradiction, assume that $\limsup_{n\to\infty} c^*(\mu_n, \mathbf{x}) > c^*(\mu', \mathbf{x})$. Then there exists $\overline{c}$ such that $\limsup_{n\to\infty} c^*(\mu_n, \mathbf{x}) > \overline{c} > c^*(\mu', \mathbf{x})$. By richness of transactions, there exists $\overline{\mu}$ such that $c^*(\overline{\mu}, \mathbf{x}) = \overline{c}$. By monotonicity of preferences in μ (i.e., in the first coordinate), $\overline{\mu} > \mu'$. However, this is a contradiction since $\mu_n < \overline{\mu}$ almost always, and therefore, by the monotonicity of preferences in μ, these transactions are almost always accepted by constant-decision type $f(\cdot) \equiv \overline{c}$ at any status quo, which is in violation of monotonicity in types. A symmetric argument applies to the case in which $\mu_n < \mu'$, and this suffices to establish the claim for any sequence.

- Q satisfies Property **WD**: If $h(\cdot) \equiv c$ rejects $\mathbf{t}'$ at status quo w, then $c^*(\mathbf{t}') \leqslant c$ (by monotonicity in types). Thus, if we also have that $c^*(\mathbf{t}) < c^*(\mathbf{t}')$, then $c^*(\mathbf{t}) < c$. Therefore, by monotonicity in types, $h(\cdot)$ rejects $\mathbf{t}$ and if $f(\cdot) \geqslant h(\cdot)$, then $f(\cdot)$ also rejects $\mathbf{t}$ at w as required.

Finally, note that if ϕ is strictly increasing and continuous, then these arguments continue to hold with respect to $\phi \circ Q(\cdot)$.

($\Longleftarrow$) Let $Q(\cdot)$ be an index satisfying all three properties. I begin by showing that $Q(\mathbf{t}) > Q(\mathbf{t}')$ if and only if $1/c^*(\mathbf{t}) > 1/c^*(\mathbf{t}')$:

- $c^*(\mu', \mathbf{x}') > c^*(\mu, \mathbf{x}) \implies Q(\mu', \mathbf{x}') \leqslant Q(\mu, \mathbf{x})$: Follows from Property **WD** by monotonicity in types (setting $h \equiv c^*(\mu, \mathbf{x})$ and $f(\cdot) \equiv c^*(\mu', \mathbf{x}')$).

- $c^*(\mu', \mathbf{x}') > c^*(\mu, \mathbf{x}) \implies Q(\mu', \mathbf{x}') \neq Q(\mu, \mathbf{x})$: Assume towards contradiction that $c^*(\mu', \mathbf{x}') > c^*(\mu, \mathbf{x})$ and $Q(\mu', \mathbf{x}') = Q(\mu, \mathbf{x})$. Then by richness of transactions, there exists a small $\varepsilon > 0$ such that $(\mu + \varepsilon, \mathbf{x}) \in T$ and by richness of preferences, $c^*(\mu', \mathbf{x}') > c^*(\mu + \varepsilon, \mathbf{x})$. Additionally, by Property **M**, $Q(\mu', \mathbf{x}') = Q(\mu, \mathbf{x}) > Q(\mu + \varepsilon, \mathbf{x})$. A contradiction (to the previous bullet).

- $c^*(\mu', \mathbf{x}') = c^*(\mu, \mathbf{x}) \implies Q(\mu', \mathbf{x}') = Q(\mu, \mathbf{x})$: Assume towards contradiction that $c^*(\mu', \mathbf{x}') = c^*(\mu, \mathbf{x})$ but that, without loss of generality, $Q(\mu', \mathbf{x}') > Q(\mu, \mathbf{x})$. Then by richness of preferences and of transactions and by Property **C**, there exists a small $\varepsilon > 0$ such that $(\mu' + \varepsilon, \mathbf{x}') \in T$, and in addition, $c^*(\mu' + \varepsilon, \mathbf{x}') > c^*(\mu, \mathbf{x})$ and $Q(\mu' + \varepsilon, \mathbf{x}') > Q(\mu, \mathbf{x})$. A contradiction (to the first bullet).

This completes the proof that $Q(\cdot)$ and $1/c^*(\cdot)$ are ordinally equivalent. Hence, there exists an increasing $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q(\cdot) = \phi(1/c^*(\cdot))$. It remains to be shown that $\phi(\cdot)$ is continuous. To see this, let $(\mu, \mathbf{x}) \in T$ and fix $\mathbf{x}$. Then, since $1/c^*(\cdot, \mathbf{x})$ is continuous in μ and has full support, if ϕ were discontinuous, then $Q(\cdot, \mathbf{x})$ would also be. However, this would violate Property **C**. □

3.2 Application: Riskiness of Additive Gambles

An *index of riskiness* is a function $Q : \mathcal{G} \to \mathbb{R}_+$ that associates each gamble with a positive real. For example, the *Aumann–Serrano index of riskiness* of the gamble $\mathbf{g}$, $Q^{AS}(\mathbf{g})$, is implicitly defined by the equation

$$\mathbb{E}\left[\exp\left(-\frac{\mathbf{g}}{Q^{AS}(\mathbf{g})}\right)\right] = 1,$$

and the *Foster–Hart index of riskiness* of the gamble $\mathbf{g}$, $Q^{FH}(\mathbf{g})$, is implicitly defined by the equation

$$\mathbb{E}\left[\log\left(1+\frac{\mathbf{g}}{Q^{FH}(\mathbf{g})}\right)\right]=0.$$

Note that an index of riskiness is *objective* in the sense that its value depends only on the gamble and not on any agent-specific attribute.

By Theorem 1, the following axioms pin down uniquely the Aumann–Serrano index of riskiness:

- **Weak monotonicity.** For every gamble $\mathbf{g}\in\mathcal{G}$ and any $\varepsilon>0$, if $\mathbf{g}+\varepsilon\in\mathcal{G}$, then $Q(\mathbf{g})>Q(\mathbf{g}+\varepsilon)$.
- **Weak continuity.** For every gamble $\mathbf{g}\in\mathcal{G}$ and a sequence of gambles $\mathbf{g}+\varepsilon_n$, if $\lim_{n\to\infty}\varepsilon_n=0$, then $\lim_{n\to\infty}Q(\mathbf{g}+\varepsilon_n)=Q(\mathbf{g})$.
- **Weak duality.** Let $\mathbf{g},\mathbf{g}'\in\mathcal{G}$ be a pair of gambles such that $Q(\mathbf{g})>Q(\mathbf{g}')$. Let u,v be CARA agents with $\rho_v(\cdot)\geqslant\rho_u(\cdot)$. Then for any wealth w, if u rejects $\mathbf{g}'$ at w, then v rejects $\mathbf{g}$ at w.

Corollary 1. *An index of riskiness Q satisfies weak monotonicity, weak continuity, and weak duality if and only if there exists a continuous and strictly increasing function $\phi:\mathbb{R}_{++}\to\mathbb{R}_{++}$ such that $Q\equiv\phi\circ Q^{AS}$.*

The axioms presented above are labeled "weak" since they are weaker versions of the ones found in Aumann and Serrano (2008, Theorem D). Specifically, whereas Aumann and Serrano require monotonicity in first-order stochastic dominance,[9] weak monotonicity only requires that a gamble that pays \$1 more in every state of the world be deemed less risky (more favorable) by the index. Weak continuity is also less demanding than Aumann and Serrano's continuity requirement. For example, the Foster–Hart index is weakly continuous, but it does not satisfy Aumann and Serrano's continuity requirement (Hart, 2011). Finally, relative to the duality axiom, weak duality—which ensures that the index satisfies Property **WD**—imposes the monotonicity of decisions in the value of the index only for a subset of pairs of types. Corollary 1 therefore strengthens Aumann and Serrano's theorem.

3.3 Application: The Delay of Investment Cashflows

Setting. This section considers the classic capital budgeting problem. An *investment cashflow* requires an investment of $\psi>0$ at time t in exchange for a stream of income that will arrive in the future. I denote the set of all possible streams by $\mathcal{S}$ with a typical element $\mathbf{s}=(s_n,t_n)_{n=1}^{N(\mathbf{s})}$, where s_n and t_n are both positive for each n. The interpretation is that s_n dollars will be paid to the investor at time $t+t_n$. The set of investment cashflows, $\mathcal{C}$, consists of pairs $(\psi,\mathbf{s})$ such that $\psi>0$, $\mathbf{s}=(s_n,t_n)_{n=1}^{N(\mathbf{s})}\in\mathcal{S}$, and $\sum_{n=1}^{N(\mathbf{s})}s_n>\psi$. To economize on notation, I henceforth write N rather than $N(\mathbf{s})$ or $N(\mathbf{c})$, noting that it remains cashflow-specific (and not uniformly bounded).

Agents (individuals or social planners) have positive time preference: for any $\Delta>0$, they prefer a dollar at time t to a dollar at time $t+\Delta$. An agent i evaluates cashflows using a continuous schedule

[9] A gamble $\mathbf{g}$ *first-order stochastically dominates* $\mathbf{g}'$ if and only if for every weakly increasing (not necessarily concave) utility function u and every $w\in\mathbb{R}$, $\mathbb{E}[u(w+\mathbf{g})]\geqslant\mathbb{E}[u(w+\mathbf{g}')]$ with strict inequality for at least one such function. A gamble $\mathbf{g}$ *second-order stochastically dominates* $\mathbf{g}'$ if and only if for every weakly increasing weakly concave utility function u and every $w\in\mathbb{R}$, $\mathbb{E}[u(w+\mathbf{g})]\geqslant\mathbb{E}[u(w+\mathbf{g}')]$ with strict inequality for at least one such function.

of positive instantaneous discount rates, $r_i(t)$. As a result, the *net present value (NPV)* of an investment cashflow $\mathbf{c} \equiv \left(\psi, (s_n, t_n)_{n=1}^N\right)$ for agent i at time t is

$$NPV(\mathbf{c}, i, t) := -\psi + \sum_{n=1}^{N} e^{-\int_t^{t+t_n} r_i(z)dz} s_n.$$

Agent i *accepts* cashflow $\mathbf{c}$ at time t if $NPV(\mathbf{c}, i, t) > 0$ and *rejects* it otherwise.

Mapping to the General Model. This model is a special case of the general model. First, the behavior of an agent i is fully pinned down by $r_i(\cdot)$ and $t \in \mathbb{R}$. Furthermore, $r_i(\cdot)$ is positive and continuous. Additionally, the decisions of constant discounting rate (CDR) agents (ones with $r_i(\cdot) \equiv d$) are time-invariant—they make the same decisions at every point in time. Next, setting $T = \{(\mu, \mathbf{s}) \mid (1/\mu, \mathbf{s}) \in \mathcal{C}\}$, the mapping $(\psi, \mathbf{s})$ to $(1/\psi, \mathbf{s})$ is a bijection between the set $\mathcal{C}$ and T (this transformation ensures that μ, is desirable, unlike ψ). Monotonicity in μ follows from the monotonicity of NPV with respect to time dominance (Bøhren and Hansen, 1980; Ekern, 1981). The condition as stated is met thanks to the assumption that agents reject transactions when they are indifferent. Monotonicity in types follows from Lemma A.1.[10] For richness of transactions, note first that if $(\psi, \mathbf{s}) \in \mathcal{C}$, then for sufficiently small $\varepsilon > 0$, whenever $|1/\psi - 1/\psi'| < \varepsilon$, we have $(\psi', \mathbf{s}) \in \mathcal{C}$. Richness of preferences and the second part of richness of transactions follow from Lemma A.2.

Inverse IRR Index of Delay. An *index* of delay is a function $Q : \mathcal{C} \to \mathbb{R}$. The fact that Q does not depend on t makes it a time expression that does not depend on the start date, like "in a week," as opposed to expressions like "this Tuesday" whose interpretation depends on whether they are said on Friday or Monday. This restriction can be interpreted as an axiom.

The *internal rate of return* (IRR, Fisher, 1930) of the investment cashflow $\mathbf{c}$ is the unique positive solution $\alpha^*(\mathbf{c})$ of the equation

$$-\psi + \sum_{n=1}^{N} e^{-\alpha t_n} s_n = 0.$$

Existence and uniqueness follow from Lemma A.2, which generalizes Norstrøm (1972). For any investment cashflow $\mathbf{c}$, the *inverse IRR index of delay*, Q^D, is equal to the inverse of $\alpha^*(\mathbf{c})$:

$$Q^D(\mathbf{c}) := \frac{1}{\alpha^*(\mathbf{c})}.$$

By Theorem 1, the following three axioms pin down uniquely the inverse IRR index of delay:

- **Monotonicity in investment.** For any $(\psi, \mathbf{s})$ and $(\psi', \mathbf{s})$ in $\mathcal{C}$, if $\psi > \psi'$, then $Q(\psi, \mathbf{s}) > Q(\psi', \mathbf{s})$.

- **Continuity in investment.** For every transaction $(\psi, \mathbf{s}) \in \mathcal{C}$ and a sequence of transactions $(\psi + \varepsilon_n, \mathbf{s})$, if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} Q(\psi + \varepsilon_n, \mathbf{s}) = Q(\psi, \mathbf{s})$.

[10] The use of acceptance and rejection allows me to avoid the *reswitching* problem of the *Cambridge capital controversy* (see Cohen and Harcourt, 2003). Although in general, choices between two cashflows may not be monotonic in agents' discounting rate, Lemma A.1 shows that acceptance and rejection decisions of investment cashflows are monotonic in discounting rates.

- **Weak duality.** Let $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$ be a pair of cashflows such that $Q(\mathbf{c}) > Q(\mathbf{c}')$. Let i, j be CDR agents with $r_j(\cdot) \geqslant r_i(\cdot)$. Then for any time t, if i rejects $\mathbf{c}'$ at t, then j rejects $\mathbf{c}$ at t.

Corollary 2. *An index of delay Q satisfies monotonicity, continuity, and weak duality if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^D$.*

4 Local Q-Aversion

In this section, I augment the framework with additional structure: some transactions will be considered "small" and decisions on small transactions will be determined only "locally." This structure allows me to consider local tastes to the property measured by the index. I derive conditions under which these tastes are monotonic in $f(w)$ (where an agent is described by the pair $(f(\cdot), w)$).

4.1 Enriching the Framework

I augment the framework with the following assumptions:

Small transactions. Denote by $T_\varepsilon \subseteq T$ the set of transactions of "size" less than ε. I assume that $T_\varepsilon \subseteq T_{\varepsilon'}$ whenever $\varepsilon' > \varepsilon$.

Locality of small transactions. For any pair of agents $(f(\cdot), w)$ and $(g(\cdot), w)$, if $f|_{(w-\delta,w+\delta)} \equiv g|_{(w-\delta,w+\delta)}$ for some $\delta > 0$, then there exists $\varepsilon > 0$ such that for any $\mathbf{t} \in T_\varepsilon$, the agent $(f(\cdot), w)$ accepts $\mathbf{t}$ if and only if $(g(\cdot), w)$ accepts $\mathbf{t}$.

Richness of small transactions. For any $\varepsilon > 0$ and $c > 0$, there exists $\mathbf{t} \in T_\varepsilon$ such that $c^*(\mathbf{t}) = c$.

Let $R_Q^\varepsilon(f(\cdot), w)$ denote the highest value that Q assigns to a transaction in T_ε that $(f(\cdot), w)$ accepts. Formally,

$$R_Q^\varepsilon(f(\cdot), w) := \sup \{Q(\mathbf{t}) | \mathbf{t} \in T_\varepsilon \text{ and } (f(\cdot), w) \text{ accepts } \mathbf{t}\}.$$

Similarly, let $S_Q^\varepsilon(f(\cdot), w)$ denote the lowest value that Q assigns to a transaction in T_ε that $(f(\cdot), w)$ rejects. Formally,

$$S_Q^\varepsilon(f(\cdot), w) := \inf \{Q(\mathbf{t}) | \mathbf{t} \in T_\varepsilon \text{ and } (f(\cdot), w) \text{ rejects } \mathbf{t}\}.$$

Denote by $R_Q^0(f(\cdot), w)$ and $S_Q^0(f(\cdot), w)$ the limits of their respective values as ε decreases to zero.[11] Finally, we say that $(f(\cdot), w)$ is *locally at least as averse to Q* as $(g(\cdot), w')$ if $R_Q^0(f(\cdot), w) \leqslant S_Q^0(g(\cdot), w')$. Informally, this definition requires that the Q-highest local transaction that $(f(\cdot), w)$ accepts is ranked by the index Q lower than the Q-lowest local transaction that $(g(\cdot), w')$ rejects.

[11] The existence of a limit in the wide sense is guaranteed since $T_\varepsilon \subseteq T_{\varepsilon'}$ whenever $\varepsilon < \varepsilon'$. By the positivity of Q, these limits may take values in $[0, \infty]$.

4.2 The Relationship Between Preferences and Local Q-Aversion

It is not difficult to construct indices such that $R^0_Q(f(\cdot), w) = \infty$ and $S^0_Q(f(\cdot), w) = 0$ for some $(f(\cdot), w)$. However, such indices are not useful in simplifying the decision-making process, in the sense that even when the transaction is small, the agent cannot simply rely on the index and a simple cutoff value (e.g., by only investing in bonds rated AA or higher). The following property, local consistency, requires that agents can act on such rules, at least when the transaction is small.

Definition 4. *An index Q satisfies **Property LC** if for every decision maker $(f(\cdot), w)$, for every $\delta > 0$, there exists $\lambda \in \mathbb{R}_{++}$ and $\varepsilon > 0$ such that any transaction $\mathbf{t}$ in T_ε with $Q(\mathbf{t}) > \lambda + \delta$ is rejected by $(f(\cdot), w)$ and any transaction $\mathbf{t}$ in T_ε with $Q(\mathbf{t}) < \lambda - \delta$ is accepted by $(f(\cdot), w)$.*

Lemma 1 follows immediately from this definition.

Lemma 1. *If an index Q satisfies Property **LC**, then $R^0_Q(f(\cdot), w) \leqslant S^0_Q(f(\cdot), w)$ for any decision maker $(f(\cdot), w)$.*

Theorem 2 shows that if Q satisfies Property **LC**, then the local aversion to Q of agent $f(\cdot)$ at status quo w is determined by the value $f(w)$, where agents with higher values are at least as averse to Q as agents with lower values.

Theorem 2. *For any $(f(\cdot), w)$ and $(g(\cdot), w')$ such that $f(w) < g(w')$, if Q satisfies Property **LC**, then $(g(\cdot), w')$ is at least as averse to Q as $(f(\cdot), w)$.*

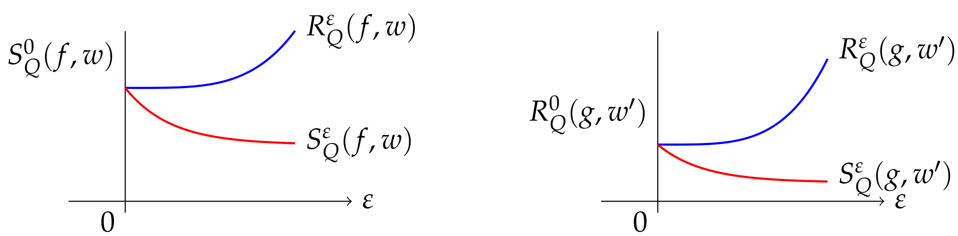

*Figure 2: Decisions over small transactions under Properties **LC** and **F** when $f(w) < g(w')$.*

Proof. Since $f(w) < g(w')$, we have that $c := (f(w) + g(w'))/2$ satisfies $f(w) < c < g(w)$. By continuity of f and g, there exists $\delta > 0$ such that for any $\hat{w} \in (w - \delta, w + \delta)$ and $\hat{w}' \in (w' - \delta, w' + \delta)$, we have $f(\hat{w}) < c < g(\hat{w}')$. The locality of small transactions attributes to $(f(\cdot), w)$, $(g(\cdot), w')$ and the constant-decision type c at both status quo values an $\bar{\varepsilon}$-environment in which decisions are based only on the values of the function in $(w - \delta, w + \delta)$. Let $\varepsilon > 0$ be smaller than all these $\bar{\varepsilon}$'s. By locality of small transactions and monotonicity in types, $(g(\cdot), w')$ rejects any transaction in T_ε that the constant-decision type c rejects at w'. Since constant-decision types make the same decisions at every status quo, the same holds for the constant-decision type c at status quo w. Furthermore, by locality of small transactions and monotonicity in types, the constant-decision type c at status quo w rejects any transaction in T_ε that $(f(\cdot), w)$ rejects. Hence, $R^{\varepsilon'}_Q(g(\cdot), w') \leqslant R^{\varepsilon'}_Q(f(\cdot), w)$ for any $\varepsilon' < \varepsilon$, and so $R^0_Q(g(\cdot), w') \leqslant R^0_Q(f(\cdot), w)$. To complete the proof, note that by Lemma 1 we have $R^0_Q(f(\cdot), w) \leqslant S^0_Q(f(\cdot), w)$. $\square$

Before proceeding to the final step of the analysis, I present an additional property that guarantees that $R_Q^0 \equiv S_Q^0$. This property is held by many indices across the settings I study. When $R_Q^0 \equiv S_Q^0$, it is reasonable to think of R_Q^0 as a numerical index over local preferences for Q (not over transactions). In applications, I show that this measure coincides with standard notions of local preferences that do not rely on an index. For example, the local aversion to Aumann–Serrano riskiness coincides with Arrow–Pratt absolute risk aversion.

Definition 5. *An index Q satisfies* ***Property* F** *if for every $\varepsilon > 0$, the image of T_ε under Q is full (i.e., equals $\mathbb{R}_{++}$).*

Richness of small transactions guarantees that indices of the form $1/c^*(\mathbf{t})$ satisfy Property **F**.

Lemma 2. *If Q satisfies Property* **F**, *then $R_Q^0(f(\cdot), w) \geqslant S_Q^0(f(\cdot), w)$ for any decision maker $(f(\cdot), w)$.*

Proof. By Property **F**, the image of T_ε under Q is full. Furthermore, the union of the set of accepted transactions and the set of rejected transactions equals the set of all transactions. Therefore, for any decision maker, the supremum of Q over the set of accepted transactions in T_ε (i.e., R_Q^ε) is at least as high as the infimum of Q over the set of rejected transactions in T_ε (i.e., S_Q^ε) for any ε. □

When $R_Q^0 \equiv S_Q^0$, it is natural to think of $1/R_Q^0$ and $1/S_Q^0$ as measures of local aversion to the property measured by the index Q.[12] In the applications I present, indices of the form $Q(\mathbf{t}) := 1/c^*(\mathbf{t})$ satisfy Properties **F** and **LC**, and so they also satisfy that $R_Q^0 \equiv S_Q^0$, and this implies that the same holds for their continuous increasing transformations.

Illustration: Additive Gambles. I begin by defining the set of ε-size gambles:

$$\mathcal{G}_\varepsilon := \{\mathbf{g} \in \mathcal{G} \mid \max\{L(\mathbf{g}), M(\mathbf{g})\} < \varepsilon\}.$$

I now show that with this addition, the additive gambles setting is a special case of my (enriched) framework. First, if $\varepsilon' < \varepsilon$, then $\mathcal{G}_{\varepsilon'} \subseteq \mathcal{G}_\varepsilon$. Furthermore, locality of small transactions is satisfied since if $\rho_u(\cdot) = \rho_v(\cdot)$ in an ε-environment of w, then v can be normalized (through an affine transformation) to coincide with u in this environment. Finally, for richness of small transactions, note that for any $c > 0$ and $\varepsilon > 0$ the gamble $\mathbf{g}^{\varepsilon,c} = [\varepsilon, \frac{e^{c\varepsilon}}{1+e^{c\varepsilon}}; -\varepsilon, \frac{1}{1+e^{c\varepsilon}}]$ is accepted by a CARA-ρ agent if and only if $\rho < c$.

Since $Q^{AS}(\mathbf{g}^{\varepsilon,c}) = 1/c$, gambles of this form show that $Q^{AS}(\cdot)$ has Property **F**. Finally, to see that $Q^{AS}(\cdot)$ has Property **LC**, note that for any u and w, for any $\delta \in (0, \rho_u(w))$, there exists $\varepsilon > 0$ such that for any $w' \in (w - \varepsilon, w + \varepsilon)$, we have $\rho_u(w') \in (\rho_u(w) - \delta, \rho_u(w) + \delta)$. Property **LC** therefore follows by locality of small transactions, monotonicity in types, and the fact that a CARA agent accepts a gamble if and only if the gamble's Aumann–Serrano riskiness is lower than the inverse of the agent's ARA.

Corollary 3. *For any pair of utilities, u, v, and pair of wealth levels, w, w', if $\rho_u(w) > \rho_v(w')$, then u at w is at least as averse to Aumann–Serrano-riskiness as v at w'.*

Lemma A.5 in the appendix shows that an agent u with wealth w is at least as averse to Aumann–Serrano riskiness as an agent v with wealth w' if and only if $\rho_u(w) \geqslant \rho_v(w')$.

[12] I use the notational convention that "$1/0 = \infty$" and "$1/\infty = 0$."

Illustration: Capital Budgeting. I begin by defining the set of ε-size cashflows:

$$\mathcal{C}_\varepsilon := \left\{ \mathbf{c} \in \mathcal{C} \mid \mathbf{c} = \left(\psi, (s_n, t_n)_{n=1}^N \right) \text{ and } \max\{t_n\} < \varepsilon \right\}.$$

I now show that with this addition, the capital budgeting setting is a special case of the enriched framework. First, if $\varepsilon' < \varepsilon$, then $\mathcal{C}_{\varepsilon'} \subseteq \mathcal{C}_\varepsilon$. Furthermore, locality of small transactions is satisfied since if $r_i(\cdot) \equiv r_j(\cdot)$ in an ε-environment of t, then i and j attribute the same NPV to any ε-size cashflow beginning at t. Finally, for richness of small transactions, note that for any $r > 0$ and $\varepsilon > 0$, the cashflow $\mathbf{c}^{\varepsilon,r} = (1, (\exp(\varepsilon r), \varepsilon))$ is accepted by a CDR-δ agent if and only if $\delta < r$.

Since $Q^D(\mathbf{c}^{\varepsilon,r}) = 1/r$, cashflows of this form show that $Q^D(\cdot)$ has Property **F**. Finally, to see that $Q^D(\cdot)$ has Property **LC**, note that for any i and t, for any $\delta \in (0, r_i(t))$, there exists $\varepsilon > 0$ such that for any $t' \in (t-\varepsilon, t+\varepsilon)$, we have $r_i(t') \in (r_i(t) - \delta, r_i(t) + \delta)$. Property **LC** therefore follows by locality of small transactions, monotonicity in types, and the fact that a CDR agent accepts a cashflow if and only if its Q^D-delay is lower than the inverse of the agent's discounting rate.

Corollary 4. *For any pair of agents, i, j, and pair of times t, t', if $r_i(t) > r_j(t')$, then i at t is at least as averse to Q^D-delay as j at t'.*

Lemma A.6 in the appendix shows that an agent i at time t is at least as averse to Q^D-delay as agent j at t' if and only if $r_i(t) \geqslant r_j(t')$.

5 Connecting Small and Large Transactions

Samuelson (1963) shows that *"if you would always refuse to take favorable odds on a single toss, you must rationally refuse to participate in any (finite) sequence of such tosses."* However, Samuelson also warns against undue extrapolation of his theorem, saying, *"It does not say that one must always refuse a sequence if one refuses a single venture: if, at higher income levels the single losses become acceptable, and at lower levels the penalty of losses does not become infinite, there might well be a long sequence that it is optimal."* This discussion motivates the following requirement from an index.

Definition 6. *An index satisfies **Property GS** if for any $f(\cdot)$ and w, we have*

$$S_Q^\infty(f(\cdot), w) \geqslant \inf_{w'} S_Q^0(f(\cdot), w') \;\; \textit{and} \;\; R_Q^\infty(f(\cdot), w) \leqslant \sup_{w'} R_Q^0(f(\cdot), w').$$

Property **GS** links the value of the index on large transactions to its value on small transactions. It requires that (1) no agent accepts a large transaction of a certain level of Q if he rejects small transactions of the same degree of Q no matter the status quo, and (2) no agent rejects a large transaction of a certain degree of Q if he accepts small ones of the same degree of Q no matter the status quo. One rationale for this requirement is that large transactions can be constructed by compounding many small transactions and so if the index guides the decision maker to always accept (reject) the entire collection of small transactions, it should make the same recommendation with respect to the composition of these transactions.[13]

[13]For example, Foster and Hart (2013) show that any gamble can be approximated arbitrarily well by exposing the agent

Relative to Property **WD**, Property **GS** has an appealing feature: it does not rely on interpersonal comparisons. However, Theorem 3 shows that for indices that satisfy Property **LC**, Property **GS** implies Property **WD** (i.e., Property **GS** is a stronger requirement than Property **WD**).[14]

Theorem 3. *Any index Q satisfying Properties **LC** and **GS** satisfies Property **WD**.*

Proof. Let $h(\cdot) \equiv c$ and $f(\cdot) \equiv c' \geqslant c$ be two fixed decision types with status quo w. Let $\mathbf{t}, \mathbf{t}' \in T$ be a pair of transactions such that $h(\cdot)$ rejects $\mathbf{t}$ at w and $f(\cdot)$ accepts $\mathbf{t}'$ at w. We need to show that $Q(\mathbf{t}') \leqslant Q(\mathbf{t})$.

Since $f(\cdot)$ and $h(\cdot)$ are constant-decision types,

$$\sup_{w'} R_Q^0 \left(f(\cdot), w'\right) = R_Q^0 \left(f(\cdot), w\right)$$

and

$$\inf_{w'} S_Q^0 \left(h(\cdot), w'\right) = S_Q^0 \left(h(\cdot), w\right).$$

Hence, by Lemma 1 and Theorem 2,

$$\sup_{w'} R_Q^0 \left(f(\cdot), w'\right) \leqslant \inf_{w'} S_Q^0 \left(h(\cdot), w'\right).$$

By Property **GS**, this implies

$$R_Q^\infty \left(f(\cdot), w\right) \leqslant \sup_{w'} R_Q^0 \left(f(\cdot), w'\right) \leqslant \inf_{w'} S_Q^0 \left(h(\cdot), w'\right) \leqslant S_Q^\infty \left(h(\cdot), w\right).$$

By definition, since $f(\cdot)$ accepts $\mathbf{t}'$, we have

$$Q(\mathbf{t}') \leqslant R_Q^\infty(f(\cdot), w)$$

and since $h(\cdot)$ rejects $\mathbf{t}$, we have

$$S_Q^\infty(h(\cdot), w) \leqslant Q(\mathbf{t}).$$

Altogether we obtain

$$Q(\mathbf{t}') \leqslant R_Q^\infty(f(\cdot), w) \leqslant S_Q^\infty(h(\cdot), w) \leqslant Q(\mathbf{t}),$$

establishing that Q satisfies Property **WD**. □

5.1 Application: Riskiness of Additive Gambles

By Theorem 3, the following two axioms imply the weak duality axiom:

- **Local consistency.** For any u and w, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent u

to the same small gamble over and over, stopping once reaching certain wealth levels. Similarly, an investment cashflow with IRR r can be approximated arbitrarily well by a composition of identical short horizon cashflows (with different starting times), each having IRR of r.

[14]The proof holds for any index such that the relation "at least as averse to Q" is reflexive. By Theorem 2, this includes indices that satisfy Property **LC**.

with wealth w accepts any gamble $\mathbf{g} \in \mathcal{G}_\varepsilon$ with $Q(\mathbf{g}) < \lambda - \delta$ and rejects any gamble $\mathbf{g} \in \mathcal{G}_\varepsilon$ with $Q(\mathbf{g}) > \lambda + \delta$.

- **Generalized Samuelson.** For any u and w, for any level of Q-riskiness $c > 0$:

  1. if for every w' there exists $\varepsilon > 0$ such that u accepts at w' any gamble $\mathbf{g} \in \mathcal{G}_\varepsilon$ with $Q(\mathbf{g}) \leqslant c$, then u accepts any gamble $\mathbf{g}' \in \mathcal{G}$ with $Q(\mathbf{g}') \leqslant c$.
  2. if for every w' there exists $\varepsilon > 0$ such that u rejects at w' any gamble $\mathbf{g} \in \mathcal{G}_\varepsilon$ with $Q(\mathbf{g}) \geqslant c$, then u rejects any gamble $\mathbf{g}' \in \mathcal{G}$ with $Q(\mathbf{g}') \geqslant c$.

Corollary 5. *An index of riskiness Q satisfies weak monotonicity, weak continuity, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{AS}$.*

Proof. By Theorem 3, local consistency and the generalized Samuelson property imply weak duality; thus, by Corollary 1, Q^{AS} and its continuous increasing transformations are the only possible candidates. By Corollary 1, all of these indices satisfy weak monotonicity and weak continuity. Local consistency of Q^{AS} was established in Section 4.2, and this property is preserved under continuous increasing transformations. Since CARA agents with ARA of $1/c$ accept gambles if and only if their AS-riskiness is lower than c, the findings of Section 4.2 imply that the requirements of the generalized Samuelson property apply to agents that have lower (higher) ARA than $1/c$, and this follows from Jensen's inequality. This argument also applies to continuous increasing transformations of Q^{AS}. □

5.2 Application: The Delay of Investment Cashflows

By Theorem 3, the following two axioms imply the weak duality axiom:

- **Local consistency.** For any i and t, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent i at time t accepts any cashflow $\mathbf{c} \in \mathcal{C}_\varepsilon$ with $Q(\mathbf{c}) < \lambda - \delta$ and rejects any cashflow $\mathbf{c} \in \mathcal{C}_\varepsilon$ with $Q(\mathbf{c}) > \lambda + \delta$.
- **Generalized Samuelson.** For any i and t, for any level of Q-delay $d > 0$:

  1. if for every t' there exists $\varepsilon > 0$ such that i accepts at t' any cashflow $\mathbf{c} \in \mathcal{C}_\varepsilon$ with $Q(\mathbf{c}) \leqslant d$, then i accepts at t any cashflow $\mathbf{c}' \in \mathcal{C}$ with $Q(\mathbf{c}') \leqslant d$.
  2. if for every t' there exists $\varepsilon > 0$ such that i rejects at t' any cashflow $\mathbf{c} \in \mathcal{C}_\varepsilon$ with $Q(\mathbf{c}) \geqslant d$, then i rejects at t any cashflow $\mathbf{c}' \in \mathcal{C}$ with $Q(\mathbf{c}') \geqslant d$.

Corollary 6. *An index of delay Q satisfies monotonicity, continuity, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{D}$.*

Proof. By Theorem 3, local consistency and the generalized Samuelson property imply weak duality; thus, by Corollary 2, Q^D and its continuous increasing transformations are the only possible candidates. By Corollary 2, all of these indices satisfy weak monotonicity and weak continuity. Local consistency of Q^D was established in Section 4.2, and this property is preserved under continuous increasing transformations. Since CDR agents with a discounting rate of $1/c$ accept cashflows if and only if their Q^D-delay is lower than c, the findings of Section 4.2 imply that the requirements of the generalized Samuelson property apply to agents that have an instantaneous discounting rate lower (higher) than $1/c$, and this follows from Jensen's inequality (see Lemma A.2). This argument also applies to continuous increasing transformations of Q^D. □

6 Discussion

As discussed in the introduction, a large body of work focuses on partial orders over which all "reasonable" agents agree. Examples include first- and second-order stochastic dominance (Hanoch and Levy, 1969; Hadar and Russell, 1969; Rothschild and Stiglitz, 1970) in the context of gambles, time dominance (Bøhren and Hansen, 1980; Ekern, 1981) in the context of investment cashflows, Blackwell's (1953) order over information structures, and stochastic dominance in the presence of a risk-free asset (Levy and Kroll, 1978) for portfolio allocation problems. As the indices I derive correspond to critical constant-decision types, so long as constant-decision types fall under the definition of "reasonable" agents, the indices are monotonic with respect to these orders. Indeed, this is the case in all of the applications I study, and so each of the indices I derive is monotonic in the relevant partial orders.

Many popular indices do not have this desirable feature. Take, for example, *Value at Risk* (VaR)—a family of indices commonly used in the financial industry (Taleb, 2009). VaR indices depend on a parameter called the *confidence level*. For example, the VaR of a gamble at the 95 percent confidence level is the largest loss that occurs with probability greater than 5 percent.[15] VaR indices are not responsive to changes in the "tails," and as a result they are only weakly increasing in first-order stochastic dominance, and they may *decrease* as a result of mean-preserving spreads.

Turvey (1963) reports that in the domain of investment cashflows, *the payoff period*—the number of years it takes before the undiscounted sum of the gains realized from the investment equals its capital cost—was once commonly used by practitioners but *"[p]ractical men in industries with long-lived assets have perforce been made aware of the deficiencies of this criterion."* The payoff period suffers from deficiencies analogous to those of VaR. For example, it is not responsive to changes in the tails. In this sense, the lessons learned by investors in long-lived assets align with those learned more recently by investors in risky assets with rare tail events (Taleb, 2009).

This monotonicity property is especially appealing in the context of performance indices for portfolio allocation problems. In Appendix A.1, I derive the generalized Sharpe ratio for this environment. Many alternative performance indices, including ones inspired by Aumann and Serrano (2008), rank transactions in "unreasonable" ways. For example, the Sharpe ratio is not monotonic with respect to first-order stochastic dominance (even if $\mathbf{r}_1$ is first-order stochastically dominated by $\mathbf{r}_2$, the Sharpe ratio may rank $\mathbf{r}_1$ higher than $\mathbf{r}_2$), the inverse-AS performance index (Kadan and Liu, 2014) is not monotonic with respect to stochastic dominance in the presence of a risk-free asset,

[15]For consistency with the literature, in the present paper, I required indices to take only positive values, which rules out VaR. However, since my analysis is ordinal, one can simply take the exponent of VaR to satisfy this requirement.

and the return-to-AS-riskiness index (Aumann and Serrano, 2008; Homm and Pigorsch, 2012a) is not quasiconcave. I elaborate on performance indices and their properties in Appendix D.

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A Additional Applications of Theorem 1

A.1 A Generalized Sharpe Ratio

This section considers portfolio allocation problems where an investor faces the choice of investing his entire wealth in a risk-free asset with a guaranteed return of r_f, or paying a fee and gaining access to a risky asset, in which case the investor is free to optimally allocate his portfolio between the two assets. Agents are as in the additive gambles setting. Transactions take the form $(\psi, \mathbf{r} - r_f)$ where $\mathbf{r} - r_f \in \mathcal{G}$ is the (random) *excess return* of the risky asset, and $\psi > 0$ is the fee that must be paid to gain access to the risky asset. I denote the set of all transactions by $\mathcal{R} \cong \mathbb{R}_{++} \times \mathcal{G}$. An agent u with wealth w accepts $(\psi, \mathbf{r} - r_f)$ whenever

$$\max_{\alpha \geqslant 0} \mathbb{E}\left[u\left((w-\psi)\cdot(1+r_f) + \alpha \cdot (\mathbf{r} - r_f)\right)\right] > u(w(1+r_f)) \tag{1}$$

and rejects it otherwise.

I focus on indices of performance that depend on r_f only through $\mathbf{r} - r_f$. Under this restriction—that could be interpreted as an axiom—there is no loss of generality in assuming that $r_f = 0$.

Mapping to the general model. This model, which is adapted from Hellman and Schreiber (2018), is a special case of the general model. First, agents' behavior is fully captured by $\rho_u(w)$ and w. Thus, they can also be described by a status quo $w \in \mathbb{R}$ and all continuous functions from $\mathbb{R}$ to $\mathbb{R}_{++}$ (representing ρ_u). Additionally, CARA agents are not subject to wealth effects—they make the same decision at any wealth level. Next, setting $X := \mathcal{G}$ and $T := \mathbb{R}_{++} \times X$, the mapping $(\psi, \mathbf{g})$ to $(1/\psi, \mathbf{g})$ is a one-to-one correspondence between $\mathcal{R}$ and T (this transformation ensures that the first coordinate, μ, is desirable, unlike ψ). Monotonicity in μ follows from the monotonicity of expected utility preferences with respect to first-order stochastic dominance. The condition as stated is met thanks to the assumption that agents reject transactions when they are indifferent. Monotonicity in types follows from Jensen's inequality (see Pratt, 1964) as higher types can always mimic the portfolios of lower types. For richness of transactions, first note that if $(\psi, \mathbf{g}) \in \mathcal{R}$, then for sufficiently small $\varepsilon > 0$, whenever $|1/\psi - 1/\psi'| < \varepsilon$, we have $(\psi', \mathbf{g}) \in \mathcal{R}$. Richness of preferences and the second part of richness of transactions are simple to show using the theorem of the maximum (see, e.g., Hellman and Schreiber, 2018, Appendix C).

The generalized Sharpe ratio. An index of performance is a function $Q : \mathcal{R} \to \mathbb{R}_{++}$. Following the convention in the literature, unlike in the additive gambles setting, in this setting, higher values of the index are interpreted as higher performance (more desirable).

For each $\mathbf{g} \in \mathcal{G}$, denote by $\alpha^*_\rho(\mathbf{g})$ the optimal level of wealth that a CARA agent with ARA ρ invests in the risky asset with excess return $\mathbf{g}$, and observe that (for any w and r^f) this level is implicitly defined by

$$\mathbb{E}\left[\rho\mathbf{g} \cdot \exp\left(-\alpha^*_\rho(\mathbf{g})\rho\mathbf{g}\right)\right] = 0.$$

The index of performance Q^{GS} of the transaction $(\psi, \mathbf{g})$ is the positive solution, ρ^*, of the equation

$$\rho = -\frac{1}{\psi} \log \mathbb{E}\left[\exp\left(-\rho\alpha^*_\rho(\mathbf{g})\mathbf{g}\right)\right].$$

The index Q^{GS} is a monotonic transformation of the index S that Hellman and Schreiber (2018) develop for this setting. Hence, it is well defined and the indices are ordinally equivalent. For any fixed fee ψ, it is ordinally equivalent to the generalized Sharpe ratio (Hodges, 1998; Zakamouline and Koekebakker, 2009). I choose the formulation above to facilitate comparison across sections and because its interpretation may be more natural: it is the level of ARA that makes a CARA agent indifferent between accepting and rejecting the transaction $(\psi, \mathbf{g})$.

By Theorem 1, the following three axioms pin down Q^{GS} uniquely:

- **Monotonicity in fee.** For every $(\psi, \mathbf{g})$ and $(\psi', \mathbf{g})$ in $\mathcal{R}$, if $\psi > \psi'$, then $Q(\psi', \mathbf{g}) > Q(\psi, \mathbf{g})$.
- **Continuity in fee.** For every transaction $(\psi, \mathbf{g}) \in \mathcal{R}$ and a sequence of transactions $(\psi + \varepsilon_n, \mathbf{g})$, if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} Q(\psi + \varepsilon_n, \mathbf{g}) = Q(\psi, \mathbf{g})$.
- **Weak Duality.** Let $\mathbf{t}, \mathbf{t}' \in \mathcal{R}$ be a pair of transactions such that $Q(\mathbf{t}) > Q(\mathbf{t}')$. Let u, v be CARA agents with $\rho_v(\cdot) \geqslant \rho_u(\cdot)$. Then, for any wealth w, if u rejects $\mathbf{t}$ at w, then v rejects $\mathbf{t}'$ at w.

Corollary 7. *An index of performance Q satisfies monotonicity in fee, continuity in fee, and weak duality if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{GS}$.*

Theorem 1 of Hellman and Schreiber (2018) delivers a similar result. Corollary 7 differs from their theorem in that the monotonicity and continuity axioms correspond to the fee (rather than the excess return), and that I only require weak duality (rather than duality).

A.2 The Value of Information for Investors

This section considers the problem of information aquisition by investors (Cabrales et al., 2013, 2017). Agents' preferences are as in the additive gambles setting. Agents are facing uncertainty about the state of nature, $k \in \{1, ..., K\}$, over which they hold a full-support prior belief $q \in \Delta(K)$. They have access to Arrow–Debreu securities, traded according to the price vector $p \in \mathbb{R}^K_{++}$, with $\sum_{i=1}^K p_i = 1$. This yields the set of *investment opportunities*

$$B^* = \left\{ b \in \mathbb{R}^K | \sum_{k\in K} p_k b_k \leqslant 0 \right\}.$$

When an agent with initial wealth w chooses investment $b \in B^*$ and state k is realized, his wealth becomes $w + b_k$. Following Cabrales et al. (2017), I focus on the case that $p = q$. In this important special case, the set B^* can be interpreted as consisting of all no-arbitrage investment opportunities.

Before choosing his investment, the agent has an opportunity to engage in an *information transaction* $(\psi, \boldsymbol{\alpha})$, where $\psi > 0$ is the cost of the transaction, and $\boldsymbol{\alpha}$ is a finite-support distribution over posteriors that respects Bayes' law. Namely, denoting the probability with which the posterior $q^s \in \Delta(K)$ is realized by $q_{\boldsymbol{\alpha}}(s) > 0$, we have $\sum_s q_{\boldsymbol{\alpha}}(s) q^s = q$. Additionally, at least one full-support posterior has positive probability (information structures where each posterior excludes some state allow agents to guarantee arbitarily large profits in each state). Denote this set of information transactions by $\mathcal{A}$.[16]

[16]For comparability, note that Cabrales et al. (2017) describe the informational content of the transaction by a signal structure, and so the distribution over posteriors it induces depends on the prior through Bayes' law. Inspired by Kamenica and Gentzkow (2011), I consider distributions over posteriors, which simplifies the exposition and makes the connections with other settings clearer.

Agents optimally choose an investment opportunity in B^* given their beliefs. Therefore, the expected utility of an agent with utility u, initial wealth w and beliefs q' is

$$V(u,w,q') := \sup_{b \in B^*} \sum_k q'_k u\left(w + b_k\right).$$

Accordingly, an agent accepts the information transaction $(\psi, \boldsymbol{\alpha})$ if

$$\sum_s q_{\boldsymbol{\alpha}}(s) V(u, w-\psi, q^s) > V(u,w,q)$$

and rejects it otherwise. When $p = q$, we have that $V(u,w,q) = u(w)$, as certainty is optimal for any risk averse investor that has no informational advantage.

The normalized-value index. An *index of appeal* of information transactions is a function $Q : \mathcal{A} \to \mathbb{R}_{++}$. Following the convention in the literature, unlike in the additive gambles setting, in this setting, higher values of the index are interpreted as more informative (more desirable). The *normalized-value index of appeal*, Q^{NV}, first suggested by Cabrales et al. (2017), is defined by

$$Q^{NV}(\psi,\boldsymbol{\alpha}) := -\frac{1}{\psi}\left[\log\left(\sum_s q_{\boldsymbol{\alpha}}(s)\exp\left(-d\left(p \mid\mid q^s\right)\right)\right) + d\left(p||q\right)\right],$$

where

$$d\left(p||q\right) := \sum_k p_k \log \frac{p_k}{q_k}$$

is the *Kulback-Leibler divergence* (Kullback and Leibler, 1951). The index Q^{NV} can be interpreted as the critical level of ARA below (above) which a CARA agent accepts (rejects) the transaction (Cabrales et al., 2017). When $p = q$ we therefore have

$$Q^{NV}(\psi,\boldsymbol{\alpha}) := -\frac{1}{\psi}\left[\log\left(\sum_s q_{\boldsymbol{\alpha}}(s)\exp\left(-d\left(p \mid\mid q^s\right)\right)\right)\right].$$

Since the current model is a special case of the general model (the proof is almost identical to the portfolio allocation setting of Appendix A.1), by Theorem 1, the following three axioms pin down Q^{NV} uniquely:

- **Monotonicity in fee.** For every $(\psi, \boldsymbol{\alpha})$ and $(\psi', \boldsymbol{\alpha})$ in $\mathcal{A}$, if $\psi > \psi'$ then $Q(\psi', \boldsymbol{\alpha}) > Q(\psi, \boldsymbol{\alpha})$.
- **Continuity in fee.** For every transaction $(\psi, \boldsymbol{\alpha}) \in \mathcal{A}$ and a sequence of transactions $(\psi + \varepsilon_n, \boldsymbol{\alpha})$, if $\lim_{n\to\infty} \varepsilon_n = 0$ then $\lim_{n\to\infty} Q(\psi + \varepsilon_n, \boldsymbol{\alpha}) = Q(\psi, \boldsymbol{\alpha})$.
- **Weak Duality.** Let $\mathbf{t}, \mathbf{t}' \in \mathcal{A}$ be a pair of gambles such that $Q(\mathbf{t}) > Q(\mathbf{t}')$. Let u, v be CARA agents with $\rho_v(\cdot) \geqslant \rho_u(\cdot)$. Then, for any wealth w, if u rejects $\mathbf{t}$ at w, then v rejects $\mathbf{t}'$ at w.

Corollary 8. *An index of appeal Q satisfies monotonicity in fee, continuity in fee, and weak duality if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{NV}$.*

A.3 Riskiness of Menus of Gambles

This section considers a setting identical to the additive gambles setting with the exception that transactions correspond to a menus of finite-valued random variables from which the agent can choose. An agent u with wealth w therefore accepts the menu $\widehat{\mathbf{M}}$ if

$$\sup_{\mathbf{g}\in\widehat{\mathbf{M}}} \mathbb{E}\left[u(w+\mathbf{g})\right] > u(w)$$

and rejects it otherwise. I assume that the set of all possible transactions, $\mathcal{GM}$, consists of each menu, $\widehat{\mathbf{M}}$, such that there exists $L > 0$ so that each random variable in $\widehat{\mathbf{M}}$ takes a value weakly lower than $-L$ with positive probability. Let $L(\widehat{\mathbf{M}})$ denote the largest value of such L. I further assume that $\widehat{\mathbf{M}} \bigcap \mathcal{G} \neq \varnothing$ and $\inf_{\mathbf{g}\in\widehat{\mathbf{M}}\bigcap\mathcal{G}} Q^{AS}(\mathbf{g}) > 0$. These assumptions guarantee that no menu is accepted or rejected by all agents at all wealth levels.

A special case of a transaction in $\mathcal{GM}$ is any singleton menu consisting of a gamble $\mathbf{g} \in \mathcal{G}$. But, generally, different agents may choose *different* gambles from the menu if they accepts the transaction. For example, in the portfolio allocation problem of Appendix A.1 agents control their level of exposure to the risky asset, and in the information acquisition setting of Appendix A.2 agents can choose any investment opportunity for each realized posterior belief.

Mapping to the general model. This model is a special case of the general model. First, agents' behavior is fully captured by $\rho_u(w)$ and w. Thus, they can also be described by a status quo $w \in \mathbb{R}$ and all continuous functions from $\mathbb{R}$ to $\mathbb{R}_{++}$ (representing ρ_u). Additionally, CARA agents are not subject to wealth effects—they make the same decision at any wealth level. Next, denote $X := \left\{\widehat{\mathbf{M}} + L(\widehat{\mathbf{M}}) \mid \widehat{\mathbf{M}} \in \mathcal{GM}\right\}$ and let $T := \{(l, \mathbf{x}) \in \mathbb{R}_{++} \times X \mid \mathbf{x} - l \in \mathcal{GM}\}$. Monotonicity in μ follows from the monotonicity of expected utility preferences with respect to first-order stochastic dominance. The condition as stated is met thanks to the assumption that agents reject transactions when they are indifferent. Monotonicity in types follows from Jensen's inequality (see Pratt, 1964), since higher types can always mimic the lottery choices of lower types. For richness of transactions, first note that $L(\widehat{\mathbf{M}})$ is strictly positive for any $\mathbf{M} \in \mathcal{GM}$ and so if $(l, \mathbf{x}) \in T$ that the same holds for a small environment of l. The second part of richness of transactions and richness of preferences follow from the assumptions on $\mathcal{GM}$ by the results on additive gambles (Section 3.2).

An index of riskiness for menus of additive gambles. An *index of riskiness* for menus of additive gambles is a function $Q : \mathcal{GM} \to \mathbb{R}_{++}$. Denote

$$Q^{GM}\left(\widehat{\mathbf{M}}\right) := \inf_{\mathbf{g}\in\widehat{\mathbf{M}}\bigcap\mathcal{G}} Q^{AS}(\mathbf{g}).$$

The restrictions on the set of transactions guarantee that Q^{GM} is a well-defined index.

By Theorem 1, the following three axioms pin down Q^{GM} uniquely:

- **Weak Monotonicity.** For every menu $\widehat{\mathbf{M}} \in \mathcal{GM}$ and any $\varepsilon > 0$, if $\widehat{\mathbf{M}} + \varepsilon \in \mathcal{GM}$ then $Q(\widehat{\mathbf{M}}) > Q\left(\widehat{\mathbf{M}} + \varepsilon\right)$.

- **Weak Continuity.** For every menu $\widehat{\mathbf{M}} \in \mathcal{GM}$ and a sequence of menus, $\widehat{\mathbf{M}} + \varepsilon_n$, if $\lim_{n\to\infty} \varepsilon_n = 0$ then $\lim_{n\to\infty} Q\left(\widehat{\mathbf{M}} + \varepsilon_n\right) = Q(\widehat{\mathbf{M}})$.

- **Weak Duality.** Let $\widehat{\mathbf{M}}, \widehat{\mathbf{M}}' \in \mathcal{GM}$ be a pair of menus such that $Q(\widehat{\mathbf{M}}) > Q(\widehat{\mathbf{M}}')$. Let u, v be CARA agents with $\rho_v(\cdot) \geqslant \rho_u(\cdot)$. Then, for any wealth w, if u rejects $\widehat{\mathbf{M}}'$ at w, then v rejects $\widehat{\mathbf{M}}$ at w.

Corollary 9. *An index of riskiness for menus of additive gambles Q satisfies weak monotonicity, weak continuity, and weak duality if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{GS}$.*

Michaeli (2014, Section 3.4) contains an extension that considers a special case of the model presented here. He is interested in a situation where the agent is offered a gamble whose outcome depends on the state of the word, but the agent has multiple "plausible" priors and he is extremely optimistic in the sense that his preferences take the maximax form (Gilboa and Schmeidler, 1989). Michaeli provides a different axiomatization for (the restriction of) Q^{GM} for that setting. Of note, the main focus of Michaeli (2014) is on ambiguity *averse* agents, especially ones with utilities that take the maximin form (Gilboa and Schmeidler, 1989) or use the Hurwicz criterion (Hurwicz, 1951). His main models are also special cases of the general model, and so variants of Corollary 9 for his settings can be derived using Theorem 1, and the resulting indices coincide with the ones he derives.

A.4 Riskiness of Multiplicative Gambles

This section considers the multiplicative gambles setting. Agents' preferences are summarized by a von Neumann–Morgenstern utility function for money, $u : \mathbb{R}_{++} \to \mathbb{R}$, and a status-quo wealth, $w \in \mathbb{R}_{++}$. Utility functions are strictly increasing, strictly concave, and twice continuously differentiable. Furthermore, they satisfy

$$\varrho_u(w) := -w\frac{u''(w)}{u'(w)} > 1.$$

The function $\varrho_u(w)$ is the *Arrow–Pratt coefficient of relative risk aversion* (RRA) of u at w. The restriction on agents' RRA guarantees that they are ruin averse (Hart, 2011)—they are never willing to accept gambles that could bankrupt them.

Agents are offered a multiplicative gamble $\mathbf{r} \in \mathcal{M} := \{\mathbf{r} \in \mathcal{G} \mid \mathbb{E}[\log(1+\mathbf{r})] > 0\}$. If an agent with wealth w accepts the gamble $\mathbf{r}$, his resulting wealth is distributed according to $w(1+\mathbf{r})$. The restriction that $\mathbb{E}[\log(1+\mathbf{r})] > 0$ is equivalent to $Q^{FH}(\mathbf{r}) < 1$. It guarantees that if a gamble is taken repeatedly it does not lead the agent to ruin (Foster and Hart, 2009). A gamble $\mathbf{r}$ is *accepted* by u at wealth w if $\mathbb{E}[u(w(1+\mathbf{r}))] > u(w)$, and is *rejected* otherwise.

Mapping to the general model. This model, that was studied in Schreiber (2013) and Li (2014), is a special case of the general model. First, agents' behavior is fully captured by $\varrho_u(\cdot)$ and w. Thus, they can also be described by a status quo in $\mathbb{R}$ (describing $\log(w)$) and all continuous functions from $\mathbb{R}$ to $\mathbb{R}_{++}$ (describing $\varrho_u(\log(w)) - 1$). Additionally, CRRA agents (ones with constant $\varrho_u - 1 \equiv c > 0$) are not subject to wealth effects—they make the same decision at any wealth level. Next, the set $\mathcal{M}$ is in one-to-one correspondence with the set of pairs $(\mu, \mathbf{x})$ such that μ represents $\mathbb{E}[\log(1+\mathbf{r})]$ and $\mathbf{x}$ is a finite-valued mean-zero non-degenerate random variable representing $\log(1+\mathbf{r}) - \mathbb{E}[\log(1+\mathbf{r})]$.[17] Monotonicity in μ follows from monotonicity of expected utility preferences with respect to first-order stochastic dominance. The condition as stated is met thanks to the assumption that agents

[17] Since the set $\mathcal{M}$ is a subset of $\mathcal{G}$, I could have chosen another correspondence where μ and $\mathbf{x}$ are as in the additive gambles setting and $T := \{(\mu, \mathbf{x}) \mid \mu + \mathbf{x} \in \mathcal{M}\}$. Although this approach would "work" technically, it would require a stronger or a less natural monotonicity axiom, and so I refrain from using it.

reject transactions when they are indifferent. Monotonicity in types follows from Jensen's inequality (see Pratt, 1964). To see that the set of transactions meets the richness requirements, note that if $\mathbf{r} \in \mathcal{M}$, then for sufficiently small ε, we have $\mathbb{E}[\log(1+\mathbf{r}) \pm \varepsilon] > 0$ (since $\mathbb{E}[\log(1+\mathbf{r})] > 0$) and $\exp(\log(1+\mathbf{r}) \pm \varepsilon) - 1 \in \mathcal{G}$ (by continuity of log and exp). Richness of preferences and the second part of richness of transactions are established in Lemma A.4.

The Index of Relative Riskiness. An *index of (relative) riskiness* is a function $Q : \mathcal{M} \to \mathbb{R}_{++}$. The index of relative riskiness, Q^S, first suggested by Schreiber (2013) (see also Li, 2014), is implicitly defined by the equation $\mathbb{E}\left[(1+\mathbf{r})^{-\frac{1}{Q^S(\mathbf{r})}}\right] = 1$.

By Theorem 1, the following three axioms pin down Q^S uniquely:

- **Weak Monotonicity.** For every gamble $\mathbf{r} \in \mathcal{M}$ and any $\varepsilon > 0$, if $(1+\varepsilon)(1+\mathbf{r}) - 1 \in \mathcal{M}$ then $Q(\mathbf{r}) > Q\left((1+\varepsilon)(1+\mathbf{r}) - 1\right)$.
- **Weak Continuity.** For every gamble $\mathbf{r} \in \mathcal{M}$ and a sequence of gambles $\{(1+\varepsilon_n)(1+\mathbf{r}) - 1\}$, if $\lim_{n\to\infty} \varepsilon_n = 0$ then $\lim_{n\to\infty} Q\left((1+\varepsilon_n)(1+\mathbf{r}) - 1\right) = Q(\mathbf{r})$.
- **Weak Duality.** Let $\mathbf{r}, \mathbf{r}' \in \mathcal{M}$ be a pair of gambles such that $Q(\mathbf{r}) > Q(\mathbf{r}')$. Let u, v be CRRA agents with $\varrho_v(\cdot) \geqslant \varrho_u(\cdot)$. Then, for any wealth w, if u rejects $\mathbf{r}'$ at w, then v rejects $\mathbf{r}$ at w.

As in the additive gambles setting, the axioms are weaker versions of their counterparts in previous studies (Schreiber, 2013; Li, 2014). Hence, they strengthen their results and their justifications for duality apply.

Corollary 10. *An index of relative riskiness Q satisfies monotonicity, continuity, and weak duality if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^S$.*

To the best of my knowledge, no previous paper has considered menus of multiplicative gambles. However, the analysis of the previous section can be replicated in that case. Furthermore, the same remark holds for ambiguity averse agents as in Michaeli (2014).

B Additional Application of Theorem 2

B.1 A Generalized Sharpe Ratio

To begin with, I define the set of ε-size transactions

$$\mathcal{R}_\varepsilon := \left\{(\psi, \mathbf{g}) \in \mathcal{R} \mid \mathbf{g} \in \mathcal{G}_\varepsilon,\ Q^{AS}(\mathbf{g}) > \frac{1}{\varepsilon},\ \psi < \varepsilon\right\}.$$

Even though the set of excess returns coincides with the set of additive gambles $\mathcal{G}$, the requirement that $g \in \mathcal{G}_\varepsilon$ is, on its own, not restrictive in the portfolio allocation problem, since the agent can scale her investment level, making her indifferent between gaining access to $\mathbf{g}$ or to $\lambda\mathbf{g}$ for any $\mathbf{g}$ and $\lambda > 0$. Similarly, the requirement that $Q^{AS}(\mathbf{g}) > \frac{1}{\varepsilon}$ is, on its own, not restrictive in the portfolio allocation problem, since $Q^{AS}(\lambda\mathbf{g}) = \lambda Q^{AS}(\mathbf{g})$ for any $\mathbf{g}$ and $\lambda > 0$. However, since for small ε, $Q^{AS}(\mathbf{g})$ is approximately equal to $\mathrm{VAR}(\mathbf{g}) / \mathbb{E}[\mathbf{g}]$ for $g \in \mathcal{G}_\varepsilon$ (see Heller and Schreiber, 2020), when the two restrictions are combined, they essentially require that the agent must be exposed to substantial losses relative to her target expected return. The final requirement from transactions in $\mathcal{R}_\varepsilon$ is that their price is low.

I now show that, with this addition, this setting is a special case of the enriched framework.

Recalling that, for consistency with the literature, I assumed that higher values of the index are more desirable, I begin by adjusting the local preferences definition.[18] Given an index of performance Q, a utility function u, a wealth level w, and $\varepsilon > 0$, let

$$R_Q^\varepsilon(u, w) := \inf\{Q(\mathbf{r}) | \, \mathbf{r} \in \mathcal{R}_\varepsilon \text{ and } \mathbf{r} \text{ is accepted by } u \text{ at } w\},$$

and

$$S_Q^\varepsilon(u, w) := \sup\{Q(\mathbf{r}) | \, \mathbf{r} \in \mathcal{R}_\varepsilon \text{ and } \mathbf{r} \text{ is rejected by } u \text{ at } w\}.$$

In words, $R_Q^\varepsilon(u, w)$ is the Q-performance of the *lowest performing* accepted transaction in $\mathcal{R}_\varepsilon$ according to Q. Similarly, $S_Q^\varepsilon(u, w)$ is the Q-performance of the *highest performing* rejected transaction in $\mathcal{R}_\varepsilon$ according to Q. Of note, the supremum and the infimum are in the "reversed" direction from the general model, as higher values of the index are to be more desirable.

Accordingly, we say that u at w has *at least as much taste for Q-performance* as v at w' if for every $\delta > 0$ there exists $\varepsilon > 0$ such that $S_Q^\varepsilon(u, w) \leqslant R_Q^\varepsilon(v, w') + \delta$. Additionally, I denote $R_Q^0(u, w) := \lim_{\varepsilon \to 0^+} R_Q^\varepsilon(u, w)$ and $S_Q^0(u, w) := \lim_{\varepsilon \to 0^+} S_Q^\varepsilon(u, w)$.

I now turn to showing that this environment meets the conditions of the enriched framework. First, if $\varepsilon' < \varepsilon$ then $\mathcal{R}_{\varepsilon'} \subseteq \mathcal{R}_\varepsilon$. Furthermore, locality of small transactions follows from Lemma A.7. To show richness of small transactions consider transactions of the form $(\psi, \mathbf{g})$, where $\psi < \varepsilon$ and $\mathbf{g} = [-\varepsilon, 1/2; \varepsilon(1 + \alpha), 1/2]$. Let $c > 0$. For sufficiently low α, we have $c^*(\psi, \mathbf{g}) < c$. And since $c^*(\cdot)$ is continuously increasing in price (in fact, it is homogeneous of degree -1 in price) we can decrease ψ to $\psi' < \psi$ so that $c^*(\psi', \mathbf{g}) = c$.

The argument above also shows that $Q^{GS}(\cdot) = 1/c^*(\cdot)$ has Property **F**. Finally, $Q^{GS}(\cdot)$ has Property **LC** by locality of small transactions, monotonicity in types, and the fact that a CARA agent accepts a transaction if and only if its Q^{GS} value higher than the agent's ARA.

Corollary 11. *For any pair of utilities, u, v, and pair of wealth levels w, w', if $\rho_u(w) > \rho_v(w')$ then v at w' has at least as much taste for Q^{NV}-informativeness as u at w.*

In fact, using the "sandwiching" technique of Lemma A.5 one can show a stronger result: agent u with wealth w has at least as much taste for Q^{GS}-performance as agent v with wealth w' if and only if $\rho_u(w) \leqslant \rho_v(w')$.

B.2 The Value of Information for Investors

To begin with, I define the set of ε-size information transactions

$$\mathcal{A}_\varepsilon := \left\{(\mu, \boldsymbol{\alpha}) \in \mathcal{A} \mid \left\| q^{s'} - q^s \right\|_\infty < \varepsilon \text{ for all } s, s' \text{ s.t. } \min\left\{q_{\boldsymbol{\alpha}}(s'), q_{\boldsymbol{\alpha}}(s)\right\} > 0\right\}.$$[19]

I now show that, with this addition, this setting is a special case of the enriched framework. Recalling that for consistency with the literature I assumed that higher values of the index are more

[18] This can be avoided by considering indices for the form $1/Q$ that yield the reverse ranking relative to Q.

[19] There are other ways to define "small" transactions that are more appealing when considering the "generalized Samuelson" interpretation and the case of $p \neq q$. One approach is presented in Appendix B.3.

desirable, I begin by adjusting the local preferences definition.[20] Given an index of informativeness Q, a utility function u, a wealth level w, and $\varepsilon > 0$, let

$$R_Q^{\varepsilon}(u,w) := \inf\{Q(\mathbf{a})|\, \mathbf{a} \in \mathcal{A}_{\varepsilon} \text{ and } \mathbf{a} \text{ is accepted by } u \text{ at } w\},$$

and

$$S_Q^{\varepsilon}(u,w) := \sup\{Q(\mathbf{a})|\, \mathbf{a} \in \mathcal{A}_{\varepsilon} \text{ and } \mathbf{a} \text{ is rejected by } u \text{ at } w\}.$$

In words, $R_Q^{\varepsilon}(u,w)$ is the Q-informativeness of the *least informative* accepted transaction in $\mathcal{A}_{\varepsilon}$ according to Q. Similarly, $S_Q^{\varepsilon}(u,w)$ is the Q-informativeness of the *most informative* rejected transaction in $\mathcal{A}_{\varepsilon}$ according to Q. Of note, the supremum and the infimum are in the "reversed" direction from the general model, as higher values of the index are to be more desirable.

Accordingly, we say that u at w has *at least as much taste for Q-informativeness* as v at w' if for every $\delta > 0$ there exists $\varepsilon > 0$ such that $S_Q^{\varepsilon}(u,w) \leqslant R_Q^{\varepsilon}(v,w') + \delta$. Additionally, I denote $R_Q^0(u,w) := \lim_{\varepsilon\to 0^+} R_Q^{\varepsilon}(u,w)$ and $S_Q^0(u,w) := \lim_{\varepsilon\to 0^+} S_Q^{\varepsilon}(u,w)$.

I now turn to showing that this environment meets the conditions of the enriched framework. First, if $\varepsilon' < \varepsilon$ then $\mathcal{A}_{\varepsilon'} \subseteq \mathcal{A}_{\varepsilon}$. Furthermore, locality of small transactions follows from Lemma 4 of Cabrales et al. (2017). For richness of small transactions note that when increasing the fee ψ from zero to infinity the critical CARA agent changes continuously (in fact, it is homogeneous of degree -1), and membership in $\mathcal{A}_{\varepsilon}$ is independent of the fee.

Similarly, to show that $Q^{NV}(\cdot)$ has Property **F**, one can fix the informational content of the transaction and change ψ. Finally, $Q^{NV}(\cdot)$ has Property **LC** by locality of small transactions, monotonicity in types, and the fact that a CARA agent accepts a transaction if and only if its Q^{NV} value higher than the agent's ARA.

Corollary 12. *For any pair of utilities, u, v, and pair of wealth levels w, w', if $\rho_u(w) > \rho_v(w')$ then v at w' has at least as much taste for Q^{NV}-informativeness as u at w.*

Lemma A.8 further shows that an agent u with wealth w has at least as much taste for Q^{NV}-informativeness as agent v with wealth w' if and only if $\rho_u(w) \leqslant \rho_v(w')$.

B.3 Menus of Additive Gambles

To begin with, I define the set of ε-size menus

$$\mathcal{GM}_{\varepsilon} := \left\{\widehat{\mathbf{M}} \in \mathcal{GM} \mid \Pr(|\mathbf{g}| < \varepsilon) = 1 \text{ for all } \mathbf{g} \in \widehat{\mathbf{M}}\right\}.$$

I now show that, with this addition, the setting of menus of additive gambles is a special case of the enriched framework. First, if $\varepsilon' < \varepsilon$ then $\mathcal{GM}_{\varepsilon'} \subseteq \mathcal{GM}_{\varepsilon}$. Furthermore, locality of small transactions is satisfied as if $\rho_u(\cdot) = \rho_v(\cdot)$ in an ε-environment of w, then v can be normalized (through an affine transformation) to coincide with u in this environment. Finally, richness of small transactions is inherited from the additive gambles setting since each gamble in $\mathcal{G}$ is a singleton menu in $\mathcal{GM}$.

Since Q^{GM} coincides with Q^{AS} on singleton menus, Q^{GM} inherits Property **F** from Q^{AS} since each gamble in $\mathcal{G}_{\varepsilon}$ is a singleton menu in $\mathcal{GM}_{\varepsilon}$. Finally, to see that Q^{GM} has Property **LC**, note that for any u and w, for any $\delta \in (0, \rho_u(w))$, there exists $\varepsilon > 0$ such that for any $w' \in (w-\varepsilon, w+\varepsilon)$ we

[20] This can be avoided by considering indices for the form $1/Q$ that yield the reverse ranking relative to Q

have $\rho_u(w') \in (\rho_u(w) - \delta, \rho_u(w) + \delta)$. Property **LC** therefore follows by locality of small transactions, monotonicity in type, and the fact a CARA agent accepts a menu if and only if it contains a gamble whose Aumann–Serrano riskiness is lower than the inverse of the agent's ARA.

Corollary 13. *For any pair of utilities, u, v, and pair of wealth levels w, w', if $\rho_u(w) > \rho_v(w')$ then u at w is at least as averse to Q^{GM}-riskiness as v at w'.*

One can further show, using Lemma A.5, that an agent u with wealth w is more averse to Aumann–Serrano riskiness than agent v with wealth w' if and only if $\rho_u(w) > \rho_v(w')$.

B.4 Riskiness of Multiplicative Gambles

An analysis nearly identical to the setting of additive gambles goes through (the only difference is that RRA rather than ARA plays the key role). It follows that Q^S has Properties **F** and **LC** and that for every utility function u and every w, $1/R^0_{Q^S}(u, w) = 1/S^0_{Q^S}(u, w) = \varrho_u(w)$.

C Additional Application of Theorem 3

C.1 A Generalized Sharpe Ratio

By Theorem 3, the following two axioms imply the weak duality axiom:

- **Local Consistency.** For any u and w, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent u with wealth w rejects any transaction $\mathbf{t} \in \mathcal{R}_\varepsilon$ with $Q(\mathbf{t}) < \lambda - \delta$ and accepts any transaction $\mathbf{t} \in \mathcal{R}_\varepsilon$ with $Q(\mathbf{t}) > \lambda + \delta$.

- **Generalized Samuelson.** For any u and w, for any level of Q-performance $c > 0$:

  1. if for every w' there exists $\varepsilon > 0$ such that u accepts at w' any transaction $\mathbf{t} \in \mathcal{R}_\varepsilon$ with $Q(\mathbf{t}) \geqslant c$, then u accepts any transaction $\mathbf{t}' \in \mathcal{R}$ with $Q(\mathbf{t}') \geqslant c$.
  2. if for every w' there exists $\varepsilon > 0$ such that u rejects at w' any transaction $\mathbf{t} \in \mathcal{R}_\varepsilon$ with $Q(\mathbf{t}) \leqslant c$, then u rejects any transaction $\mathbf{t}' \in \mathcal{R}$ with $Q(\mathbf{t}') \leqslant c$.

Corollary 14. *An index of performance Q satisfies monotonicity in fee, continuity in fee, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{GS}$.*

C.2 The Value of Information for Investors

By Theorem 3, the following two axioms imply the weak duality axiom:

- **Local Consistency.** For any u and w, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent u with wealth w rejects any transaction $\mathbf{t} \in \mathcal{A}_\varepsilon$ with $Q(\mathbf{t}) < \lambda - \delta$ and accepts any transaction $\mathbf{t} \in \mathcal{A}_\varepsilon$ with $Q(\mathbf{t}) > \lambda + \delta$.

- **Generalized Samuelson.** For any u and w, for any level of Q-informativeness $c > 0$:

  1. if for every w' there exists $\varepsilon > 0$ such that u accepts at w' any transaction $\mathbf{t} \in \mathcal{A}_\varepsilon$ with $Q(\mathbf{t}) \geqslant c$, then u accepts any transaction $\mathbf{t}' \in \mathcal{A}$ with $Q(\mathbf{t}') \geqslant c$.

2. if for every w' there exists $\varepsilon > 0$ such that u rejects at w' any transaction $\mathbf{t} \in \mathcal{A}_\varepsilon$ with $Q(\mathbf{t}) \leqslant c$, then u rejects any transaction $\mathbf{t}' \in \mathcal{A}$ with $Q(\mathbf{t}') \leqslant c$.

Corollary 15. *An index of informativeness Q satisfies monotonicity in fee, continuity in fee, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{NV}$.*

C.3 Riskiness of Menus of Gambles

By Theorem 3, the following two axioms imply the weak duality axiom:

- **Local Consistency.** For any u and w, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent u with wealth w rejects any menu $\widehat{\mathbf{M}} \in \mathcal{GM}_\varepsilon$ with $Q(\widehat{\mathbf{M}}) < \lambda - \delta$ and accepts any menu $\widehat{\mathbf{M}} \in \mathcal{GM}_\varepsilon$ with $Q(\widehat{\mathbf{M}}) > \lambda + \delta$.

- **Generalized Samuelson.** For any u and w, for any level of Q-informativeness $c > 0$:

  1. if for every w' there exists $\varepsilon > 0$ such that u accepts at w' any menu $\widehat{\mathbf{M}} \in \mathcal{GM}_\varepsilon$ with $Q(\widehat{\mathbf{M}}) \geqslant c$, then u accepts any menu $\widehat{\mathbf{M}}' \in \mathcal{GM}$ with $Q(\widehat{\mathbf{M}}') \geqslant c$.
  2. if for every w' there exists $\varepsilon > 0$ such that u rejects at w' any menu $\widehat{\mathbf{M}} \in \mathcal{GM}_\varepsilon$ with $Q(\widehat{\mathbf{M}}) \leqslant c$, then u rejects any menu $\widehat{\mathbf{M}}' \in \mathcal{GM}$ with $Q(\widehat{\mathbf{M}}') \leqslant c$.

Corollary 16. *An index of riskiness Q satisfies weak monotonicity, weak continuity, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{GM}$.*

Discussion. One of the justifications for the generalized Samuelson property is that large transactions are equal, at least approximately, to the composition of many small transactions. This may not be the case in the first two applications discussed in this appendix (since such compositions may lie outside of the domain of transactions in these applications). However, since these transactions are special cases of menus of gambles, one can justify the local consistency requirement as a requirement on this larger domain.

C.4 Riskiness of Multiplicative Gambles

By Theorem 3 the following two axioms imply the weak duality axiom:

- **Local Consistency.** For any u and w, for any $\delta > 0$, there exist $\lambda > 0$ and $\varepsilon > 0$ such that agent u with wealth w accepts any gamble $\mathbf{r} \in \mathcal{M}_\varepsilon$ with $Q(\mathbf{r}) < \lambda - \delta$ and rejects any gamble $\mathbf{r} \in \mathcal{M}_\varepsilon$ with $Q(\mathbf{r}) > \lambda + \delta$.

- **Generalized Samuelson.** For any u and w, for any level of Q-riskiness $c > 0$:

  1. if for every w' there exists $\varepsilon > 0$ such that u accepts at w' any gamble $\mathbf{r} \in \mathcal{M}_\varepsilon$ with $Q(\mathbf{r}) \leqslant c$, then u accepts any gamble $\mathbf{r}' \in \mathcal{M}$ with $Q(\mathbf{r}') \leqslant c$.
  2. if for every w' there exists $\varepsilon > 0$ such that u rejects at w' any gamble $\mathbf{r} \in \mathcal{M}_\varepsilon$ with $Q(\mathbf{r}) \geqslant c$, then u rejects any gamble $\mathbf{r}' \in \mathcal{M}$ with $Q(\mathbf{r}') \geqslant c$.

Corollary 17. *An index of riskiness Q satisfies weak monotonicity, weak continuity, local consistency, and the generalized Samuelson property if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $Q \equiv \phi \circ Q^{S}$.*

D Properties of the Generalized Sharpe Ratio

In this appendix, I show that Q^{GS} has several desirable properties relative to other related indices. As the literature on performance indices is vast, I focus on approaches that are related to Aumann and Serrano (2008).

Stochastic dominance in the presence of a risk-free asset. We say that $\mathbf{r}$ first- (second-) order *stochastically dominates* $\mathbf{r}'$ *in the presence of a risk-free asset* r_f (Levy and Kroll, 1978) if for every $\alpha > 0$ there exists $\beta > 0$ such that $\alpha\mathbf{r}' + (1-\alpha) r_f$ is first- (second-) order stochastically dominated by $\beta\mathbf{r} + (1-\beta) r_f$. That is, for any portfolio one can construct using r_f and $\mathbf{r}'$, one can construct a portfolio that dominates it using r_f and $\mathbf{r}$. This partial order includes first-order stochastic dominance as a special case. As discussed in Section 6, the index Q^{GS} is monotonic with respect to stochastic dominance (of any order) in the presence of a risk-free asset, and in particular, it is homogeneous of degree zero.

Quasiconcavity. Consider two excess returns, $\mathbf{g}, \mathbf{g}' \in \mathcal{G}$, and a third excess return, $\mathbf{g}''$, that yields $\mathbf{g}$ with probability p and $\mathbf{g}'$ with probability $1-p$ (independently of $\mathbf{g}$ and $\mathbf{g}'$). An index of performance Q is *quasiconcave* if for any $\mathbf{g}, \mathbf{g}' \in \mathcal{G}$ and $p \in (0,1)$, we have $Q(\mathbf{g}'') \leqslant \max\{Q(\mathbf{g}), Q(\mathbf{g}')\}$. No decision maker strictly prefers $\mathbf{g}''$ to both $\mathbf{g}$ and $\mathbf{g}'$, but the opposite is quite common. The reason is that, generally, the agent's optimal scale of investment ($\alpha^*_{u,w}$) differs between assets, but the compound lottery $\mathbf{g}''$ requires them to commit to one scale for whichever asset will be realized. Therefore, the expected utility from the optimal scale of investment in $\mathbf{g}''$ is lower than the average (with weights p and $1-p$) of these utilities for $\mathbf{g}$ and $\mathbf{g}'$. An upshot of the discussion in Section 6 is that Q^{GS} is quasiconcave.

The Sharpe ratio. The Sharpe ratio (Sharpe, 1966) is a measure of "risk-adjusted returns" or "reward-to-variability." It is frequently used as a performance measure for portfolios (e.g., Welch, 2008). Formally, it is defined by

$$Sh\left(\mathbf{r} - r_f\right) := \frac{\mathbb{E}\left[\mathbf{r} - r_f\right]}{\sigma\left(\mathbf{r} - r_f\right)}.$$

The validity of this measure relies critically on several assumptions about the distribution of returns as well as on agents' preferences (Meyer, 1987). In particular, for general distributions, the Sharpe ratio is not monotonic with respect to first-order stochastic dominance: risky asset $\mathbf{r}_1$ may have returns that are always higher than those of asset $\mathbf{r}_2$ yet it will be ranked lower according to the index (Aumann and Serrano, 2008).[21]

Inverse AS-riskiness. Motivated by the Sharpe ratio's nonmonotonicity with respect to first-order stochastic dominance and its lack of sensitivity to high-order moments, Kadan and Liu (2014) propose a reinterpretation of the inverse of the AS index of riskiness as a performance measure and show

[21] This undesirable property of the Sharpe ratio is related to the fact that it depends only on the first two moments of the distribution. These moments are sufficient statistic for a normal distribution, and therefore basing an index on them solely may be reasonable under the assumption of normally distributed returns. However, this assumption is often rejected in settings where the Sharpe ratio is frequently used (e.g. Fama, 1965; Kat and Brooks, 2001). Moreover, a large body of literature documents the importance of higher-order moments for investment decisions (e.g. Kraus and Litzenberger, 1976; Kane, 1982; Harvey and Siddique, 2000; Barro, 2006; Gabaix, 2008). This feature makes the Sharpe ratio prone to manipulation by selling upside potential, thus creating heavy left tails (Goetzmann et al., 2007).

that it may be more favorable than the Sharpe ratio in an empirical setting. Although this index is increasing with respect to first- and second-order stochastic dominance, it is not homogeneous of degree zero (i.e., $1/Q^{AS}(\lambda\mathbf{g}) \neq 1/Q^{AS}(\mathbf{g})$). This property is normatively undesirable because for any $\lambda > 0$ and any excess return $\mathbf{g}$, the excess returns $\mathbf{g}$ and $\lambda\mathbf{g}$ allow the agent access to the exact same set of distributions over outcomes (through scaling the investment in the risky asset by $1/\lambda$).[22] Since Q^{AS} satisfies weak monotonicity and weak continuity Corollary 1, this also implies that the inverse AS-riskiness measure of performance is not monotonic with respect to first-order stochastic dominance in the presence of a risk-free asset.

Return-to-AS-riskiness. Aumann and Serrano (2008) and Homm and Pigorsch (2012a) propose a different index of performance: the expected net return divided by AS-riskiness. Formally,

$$P^{AS}(\mathbf{r} - r_f) := \frac{\mathbb{E}[\mathbf{r} - r_f]}{Q^{AS}(\mathbf{r} - r_f)}.$$

This index is not derived from first principles. Instead, it is motivated by a "reward-to-risk" reasoning where AS-riskiness takes the place of σ in the Sharpe ratio.

Like the generalized Sharpe ratio, the return-to-AS-riskiness index is ordinally equivalent to the Sharpe ratio on the domain of normal gambles (Aumann and Serrano, 2008; Schulze, 2014). Proposition A.1 shows that it is also monotonic with respect to stochastic dominance in the presence of a risk-free asset. However, Example A.1 shows that it is not quasiconcave.[23]

[22]This property makes the inverse AS-riskiness index prone to the manipulation of bundling the risky asset with the risk-free asset.

[23]As a result, the return-to-AS-riskiness index is subject to the manipulation of telling investors that their funds will be (randomly) invested in one of two portfolios. For other critiques of the mean–AS-riskiness approach, see Chew and Sagi (2022).

E Omitted Proofs

E.1 Proofs Omitted from Section 3.3 (Investment Cashflows)

Lemma A.1. *If $r_i(\cdot) \geqslant r_j(\cdot)$ and i accepts the investment cashflow $\mathbf{c} \equiv \left(\psi, (s_n, t_n)_{n=1}^N\right)$ at time t, then so does j. Furthermore, if $r_i(\cdot) > r_j(\cdot)$ then $NPV(\mathbf{c}, i, t) < NPV(\mathbf{c}, j, t)$.*

Proof. First, note that

$$NPV(\mathbf{c}, i, t) = -\psi + \sum_{n=1}^{N} e^{-\int_t^{t+t_n} r_i(z)dz} s_n,$$

and that

$$NPV(\mathbf{c}, j, t) = -\psi + \sum_{n=1}^{N} e^{-\int_t^{t+t_n} r_j(z)dz} s_n = -\psi + \sum_{n=1}^{N} e^{-\int_t^{t+t_n} r_i(z)dz} \cdot e^{-\int_t^{t+t_n} r_j(z)-r_i(z)dz} s_n.$$

Hence,

$$NPV(\mathbf{c}, j, t) - NPV(\mathbf{c}, i, t) = \sum_{n=1}^{N} e^{-\int_t^{t+t_n} r_i(z)dz} s_n \cdot \left(e^{-\int_t^{t+t_n} r_j(z)-r_i(z)dz} - 1\right)$$

The first part of the lemma follows since, for each n,

$$e^{-\int_t^{t+t_n} r_i(z)dz} s_n \geqslant 0 \text{ and } e^{-\int_t^{t+t_n} r_j(z)-r_i(z)dz} \geqslant 1.$$

The second part of the lemma follows since the inequalities are strict in that case. □

Lemma A.2. *For any investment cashflow, $\mathbf{c} \equiv \left(\psi, (s_n, t_n)_{n=1}^N\right)$, there exists a unique positive number $\alpha^*(\mathbf{c})$ such that*

$$-\psi + \sum_{n=1}^{N} e^{-\alpha^*(\mathbf{c})t_n} s_n = 0.$$

Furthermore, if $\tilde{r}(\cdot) > \alpha^*(\mathbf{c}) > \hat{r}(\cdot)$, then at each t, the agent $\tilde{r}$ accepts $\mathbf{c}$ and the agent $\hat{r}$ rejects it.

Lemma A.2 generalizes the result of Norstrøm (1972) who had shown that investment cashflows have a unique positive IRR in the discrete setting.

Proof. Define the function

$$f(\alpha) := -\psi + \sum_{n=1}^{N} e^{-\alpha t_n} s_n.$$

The function $f(\cdot)$ is continuous, and satisfies $f(0) > 0$ and $f(\alpha) < 0$ for large values of α. Hence, it has a solution by the intermediate value theorem. Lemma A.1 implies that the solution is unique, as well as the second part of the Lemma. □

E.2 Proofs Omitted from Appendix A.4 (Multiplicative Gambles)

Lemma A.3. $\mathbf{r} \in \mathcal{M} \iff \log(1 + \mathbf{r}) \in \mathcal{G}$.

Proof. In one direction, $\mathbf{r} \in \mathcal{M} \Rightarrow \mathbb{E}[\log(1+\mathbf{r})] > 0$. Hence, $\log(1+\mathbf{r})$ is well defined (finite) for any possible realization. Additionally, $\mathbf{r} \in \mathcal{M} \Rightarrow \mathbf{r} \in \mathcal{G}$, thus $\mathbf{r}$ it assumes a negative value with positive probability and therefore so does $\log(1+\mathbf{r})$. To summarize, if $\mathbf{r} \in \mathcal{M}$, then $\log(1+\mathbf{r})$ is finite-valued, has positive expectation, and takes a negative value with positive probability, hence $\log(1+\mathbf{r}) \in \mathcal{G}$

In the other direction, if $\log(1+\mathbf{r}) \in \mathcal{G}$ we have that $\log(1+\mathbf{r})$ takes a negative value with positive probability and therefore so does $\mathbf{r}$. In addition, we have $\mathbb{E}\left[\log(1+\mathbf{r})\right] > 0$. By Jensen's inequality, this implies that $\mathbb{E}[\mathbf{r}] > 0$. Hence, $\mathbf{r} \in \mathcal{M}$. □

Lemma A.4. *For any $\mathbf{r} \in \mathcal{M}$ there is a unique positive number $\alpha(\mathbf{r})$ such that $\mathbb{E}\left[(1+\mathbf{r})^{-\frac{1}{\alpha(\mathbf{r})}}\right] = 1$. Furthermore, if r corresponds to the transaction $(\mu, \mathbf{x})$ and $r' \in \mathcal{M}$ corresponds to $(\mu', \mathbf{x})$ then $\alpha(\mathbf{r}) > \alpha(\mathbf{r}')$ if and only if $\mu' > \mu$. Additionally, for any $c > 0$ there exists $\mathbf{r} \in \mathcal{M}$ with $\alpha(\mathbf{r}) = c$.*

Proof. Note that for every $\mathbf{r} \in \mathcal{M}$ and $\alpha > 0$, we have $\mathbb{E}\left[(1+\mathbf{r})^{-\frac{1}{\alpha}}\right] = \mathbb{E}\left[\exp\left(-\frac{\log(1+\mathbf{r})}{\alpha}\right)\right]$. Consequentially, Lemma A.3 and Theorem A in Aumann and Serrano (2008) imply that the unique positive solution for the equation is $Q^S(\mathbf{r}) = Q^{AS}\left(\log(1+\mathbf{r})\right)$. The monotonicity of Q^S therefore follows from the monotonicity of Q^{AS} with respect to first-order stochastic dominance, and that Q^S has a full support is inherited from Q^{AS} having this property. □

E.3 Proofs Omitted from Section 4

Lemma A.5. *For every utility function u and every w, $1/R^0_{Q^{AS}}(u,w) = 1/S^0_{Q^{AS}}(u,w) = \rho_u(w)$.*

Proof. First, observe that if u and v are two utility functions and there exists an interval $I \subseteq \mathbb{R}$ such that $\rho_u(z) \geqslant \rho_v(z)$ for every $z \in I$, then for every wealth level w and lottery $\mathbf{g}$ such that $w + \mathbf{g} \subset I$, if $\mathbf{g}$ is rejected by v at w it is also rejected by u for the same wealth level. Put differently, if $\mathbf{g}$ is accepted by u at w it is also accepted by v at the same wealth level. The reason is that the condition implies that in the domain I, u is a concave transformation of v (Pratt, 1964), hence by Jensen's inequality $u(w) \leqslant \mathbb{E}\left[u\left(w+\mathbf{g}\right)\right]$ implies that $v(w) \leqslant \mathbb{E}\left[v\left(w+\mathbf{g}\right)\right]$.

Next, recall that a *constant absolute risk aversion* (CARA) utility function with ARA coefficient of α rejects all gambles with AS-riskiness greater than $\frac{1}{\alpha}$ and accepts all gambles with AS-riskiness smaller than $\frac{1}{\alpha}$ (Aumann and Serrano, 2008). Additionally, since $u'(\cdot) > 0$, we have that $\rho_u(\cdot)$ is continuous. Specifically,

$$\forall\, \delta > 0\, \exists\, \varepsilon > 0 \text{ s.t } x \in (w-\varepsilon, w+\varepsilon) \Rightarrow \rho_u(x) \in \left(\rho_u(w) - \delta, \rho_u(w) + \delta\right). \tag{2}$$

Hence, for any $\delta < \rho_u(w)$, there exists an $\varepsilon > 0$ such that for any $\varepsilon' \in (-\varepsilon, \varepsilon)$ we have that $\rho_u(w+\varepsilon') \in (\rho_u(w) - \delta, \rho_u(w) + \delta)$ and thus $R^{\varepsilon}_{Q}(u,w) \leqslant 1/(\rho_u(w) - \delta)$ and $S^{\varepsilon}_{Q}(u,w) \geqslant 1/(\rho_u(w) + \delta)$, where I use the observation made in the first paragraph to compare u to the CARA utility functions with ARA of $\rho_u(w) + \delta$ and $\rho_u(w) - \delta$. □

Lemma A.6. *For every agent i and time t, $1/R^0_{Q^D}(u,w) = 1/S^0_{Q^D}(u,w) = r_i(t)$.*

Proof of Lemma A.6. Consider a cashflow $\mathbf{c} \in \mathcal{C}_\varepsilon$ with $\alpha^*(\mathbf{c}) = l$. By Lemma A.1, it is accepted by a CDR agent with discousting rate $l - \delta$, and rejected by a CDR agent with $l + \delta$. Since discounting functions are continuous, by the same "sandwiching" argument used in Lemma A.5, this implies that the local aversion to Q^D of an agent i at time t coincides with $r_i(t)$. □

E.4 Proofs Omitted from Appendix B

Let $\alpha^*_{u,w}(\mathbf{g})$ denote the optimal level of exposure to the risky asset of agent u with wealth w.

Lemma A.7. *For any u and w, for any $\delta > 0$ there exists $\varepsilon > 0$ such that for any $(\psi, \mathbf{g}) \in \mathcal{R}_\varepsilon$,*

$$\Pr\left\{w - \psi + \alpha^*_{u,w-\psi}(\mathbf{g})\mathbf{g} \in (w - \delta, w + \delta)\right\} = 1.$$

Proof. Fix u and w, for any $\delta > 0$. Denote $m := \min_{w' \in [w-1,w+1]} \rho_u(w')$. Since ρ_u is continuous and positive, we have $m > 0$.

Let $\varepsilon' < \min\{1/2, m/2\}$, and let $(\psi, \mathbf{g}) \in \mathcal{R}_{\varepsilon'}$. Then

$$Q^{AS}(\frac{1}{2}\mathbf{g}) = \frac{1}{2}Q^{AS}(\mathbf{g}) \geqslant 1/m$$

and so a CARA-m agent rejects the additive gamble $\frac{1}{2}\mathbf{g}$. By Jensen's inequality, so does u at $w - \psi$. Since u is strictly concave, this implies that $\alpha^*_{u,w-\psi}(\mathbf{g}) < 1/2$. To complete the proof, set $\varepsilon < \min\{\varepsilon', \delta/2\}$, guaranteeing that ψ is lower than $\delta/2$ and that the greatest loss from $\alpha^*_{u,w-\psi}(\mathbf{g})\mathbf{g}$ is lower than $1/2 \times \delta/2$. □

Lemma A.8. *For every u and w, $R_{Q^{NV}}(u,w) = S_{Q^{NV}}(u,w) = \rho_u(w)$. Furthermore, for every v and w', u at w has at least as much taste for Q^{NV}-informativeness as v at w' if and only if $\rho_u(w) \leqslant \rho_v(w')$.*

Proof. The proof is similar to the proof of Lemma A.5. Consider a sequence of transactions

$$\left\{\mathbf{a}_n = (\psi_n, \boldsymbol{\alpha}_n) \in \mathcal{A}_{\frac{1}{n}}\right\}_{n=1}^{\infty}$$

such that all of them are all accepted by u at w.

Step 1. I show that $\lim_{n\to\infty} \psi_n = 0$. Assume by way of contradiction that there is a sub-sequence of such transactions where prices do not converge to 0. Without loss of generality $\mathbf{a}_n = (\psi_n, \boldsymbol{\alpha}_n)$, and $\lim_{n\to\infty}\psi_n = \hat{\psi} \in (0, \infty]$. Let $\psi := \min\{\hat{\psi}, 1\}$. Then, there exits N such that for all $n > N$ the transactions $\mathbf{a}'_n := \left(\frac{\psi}{2}, \boldsymbol{\alpha}_n\right)$ are accepted. By Lemma 4 of Cabrales et al. (2017), as $1/n$ approaches 0, so does the scale of the optimal investment, $\|b^n\|_\infty$. Therefore, for $1/n$ small enough, $w - \frac{\psi}{2} + b^n_k$ is in a small environment of $w - \frac{\psi}{2} < w$ for all k, a contradiction.

Step 2. An implication of the discussion in Step 1 is that for any $\delta > 0$, for $1/n$ small enough, $w - \psi_n + b^n_k$ is in a δ-environment of w for all k. Since $\rho_u(w)$ is continuous, for every $\gamma > 0$ there exists a $\delta > 0$ small enough such that $z \in (w - \delta, w + \delta)$ implies $|\rho_u(z) - \rho_u(w)| < \gamma$.

Consider the CARA agents with absolute risk aversion coefficients $\rho_u(w) + \gamma$ and $\rho_u(w) - \gamma > 0$. For a small enough environment of w, I,

$$\rho_u(w) - \gamma \leqslant \inf_{z\in I}\rho_u(z) \leqslant \sup_{z\in I}\rho_u(z) \leqslant \rho_u(w) + \gamma.$$

Since Q^{NV} has Property **LC**, and since CARA agents accept a transaction if and only if its Q^{NV}-informativeness is higher than their ARA, we have $R_{Q^{NV}}(u,w) \geqslant \rho_u(w)$ and $S_{Q^{NV}}(u,w) \leqslant \rho_u(w)$. Furthermore, since Q^{NV} has Property **F**, we have $S_{Q^{NV}}(u,w) \geqslant R_{Q^{NV}}(u,w)$. Altogether we get that $S_{Q^{NV}}(u,w) = \rho_u(w) = R_{Q^{NV}}(u,w)$, as required. □

E.5 Proofs Omitted from Appendix C

Proof of Corollary 14. By Theorem 3 local consistency and the generalized Samuelson property imply weak duality, so, by Corollary 7, Q^{GS} and its continuous increasing transformations are the only possible candidates. By Corollary 7, all of these indices satisfy monotonicity in fee and continuity in fee. Local consistency of Q^{GS} was established in Appendix B.1, and this property is preserved under continuous increasing transformations. Since CARA agents with ARA of c accept a transaction if and only if its Q^{GS}-performance is higher than c, the findings of Appendix B.1 imply that the requirements of the generalized Samuelson property apply to agents that have lower (higher) ARA than c, and this follows from Jensen's inequality (since higher types can always mimic the portfolios of lower types). Finally, this property is also preserved under continuous increasing transformations. □

Proof of Corollary 15. By Theorem 3 local consistency and the generalized Samuelson property imply weak duality, so, by Corollary 8, Q^{NV} and its continuous increasing transformations are the only possible candidates. By Corollary 8, all of these indices satisfy monotonicity in fee and continuity in fee. Local consistency of Q^{NV} was established in Appendix B.2, and this property is preserved under continuous increasing transformations. Since CARA agents with ARA of c accept a transaction if and only if its Q^{NV}-performance is higher than c, the findings of Appendix B.2 imply that the requirements of the generalized Samuelson property apply to agents that have lower (higher) ARA than c, and this follows from Jensen's inequality (since higher types can always mimic the portfolios of lower types). Finally, this property is also preserved under continuous increasing transformations. □

Proof of Corollary 16. By Theorem 3 local consistency and the generalized Samuelson property imply weak duality, so, by Corollary 9, Q^{GM} and its continuous increasing transformations are the only possible candidates. By Corollary 9, all of these indices satisfy weak monotonicity and weak continuity. Local consistency of Q^{GM} was established in Appendix B.3, and this property is preserved under continuous increasing transformations. Since CARA agents with ARA of $1/c$ accept a menu if and only if its GM-riskiness is lower than c, the findings of Appendix B.3 imply that the requirements of the generalized Samuelson property apply to agents that have lower (higher) ARA than $1/c$, and this follows from Jensen's inequality (since higher types can always mimic the gamble choices of lower types). Finally, this property is also preserved under continuous increasing transformations. □

Proof of Corollary 17. By Theorem 3 local consistency and the generalized Samuelson property imply weak duality, so, by Corollary 10, Q^S and its continuous increasing transformations are the only possible candidates. By Corollary 10, all of these indices satisfy weak monotonicity and weak continuity. Local consistency of Q^S was established in Appendix B.4, and this property is preserved under continuous increasing transformations. Since CRRA agents with RRA of $1/c$ accept gambles if and only if their Q^S-riskiness is lower than c, the findings of Appendix B.4 imply that the requirements of the generalized Samuelson property apply to agents that have lower (higher) RRA than $1/c$, and this follows from Jensen's inequality. Finally, this property is also preserved under continuous increasing transformations. □

E.6 Proofs Omitted from Appendix D

Proposition A.1. *P^{AS} is monotonic with respect to first and second-order stochastic dominance in the presence of risk-free asset.*

Proof. Note that $\mathbf{r}_1$ dominates $\mathbf{r}_2$ in the presence of r_f if and only if $\mathbf{r}_1 - r_f$ dominates $\mathbf{r}_2 - r_f$ in the presence of the a risk-free asset baring interest of rate of 0, which, in turn, holds if and only if $(\mathbf{r}_1 - r_f)/\mathbb{E}[\mathbf{r}_1 - r_f]$ dominates $(\mathbf{r}_2 - r_f)/\mathbb{E}[\mathbf{r}_2 - r_f]$ in the presence of the risk-free rate of 0. Hence, it is without loss of generality to concentrate on the case that $r_f = 0$ and $\mathbb{E}[\mathbf{r}_1] = \mathbb{E}[\mathbf{r}_2] = 1$.

In this case, if $\mathbf{r}_1$ dominates $\mathbf{r}_2$ in the presence of the risk-free asset, then there exist $\alpha, \beta > 0$ such that $\alpha\mathbf{r}_1$ stochastically dominates $\beta\mathbf{r}_2$. Hence, by monotonicity of Q^{AS} with respect to first (second) order stochastic dominance, we have $Q^{AS}(\alpha\mathbf{r}_1) < Q^{AS}(\beta\mathbf{r}_2)$. By stochastic dominance we also have $\mathbb{E}[\alpha\mathbf{r}_1] \geqslant \mathbb{E}[\beta\mathbf{r}_2]$. Altogether, these inequalities imply that

$$P_0^{AS}(\mathbf{r}_1) = \frac{\mathbb{E}[\mathbf{r}_1]}{Q^{AS}(\mathbf{r}_1)} = \frac{\mathbb{E}[\alpha\mathbf{r}_1]}{Q^{AS}(\alpha\mathbf{r}_1)} > \frac{\mathbb{E}[\beta\mathbf{r}_2]}{Q^{AS}(\beta\mathbf{r}_2)} = \frac{\mathbb{E}[\mathbf{r}_2]}{Q^{AS}(\mathbf{r}_2)} = P_0^{AS}(\mathbf{r}_2)$$

as required, where the first equality on each side is by definition, the second equality uses homogeneity of degree 1 of Q^{AS} and of the expectation operator, and the inequality follows by the numerator being weakly higher and the denominator being strictly lower. □

Example A.1. *Denote*

$$\mathbf{g} = [-4.0101, 0.1429; -0.3789, 0.2715; 3.2521, 0.2504; 6.8832, 0.0545; 10.5144, 0.2807]$$

and

$$\mathbf{g}' = [-7.251, 0.0222; -0.732, 0.0286; 5.7871, 0.0583; 12.3063, 0.0842; 18.825, 0.1266; 0, 0.6801].$$

We have that $P^{AS}(\mathbf{g}) < 1.239$ and $P^{AS}(\mathbf{g}') < 1.239$.

Let $\mathbf{g}''$ denote an excess return that is distributed according to $\mathbf{g}$ with probability half and otherwise according to $\mathbf{g}'$, independently of $\mathbf{g}$ and $\mathbf{g}'$. Formally,

$$\mathbf{g}'' = [-4.0101, 0.5\cdot 0.1429; -0.3789, 0.5\cdot 0.2715; 3.2521, 0.5\cdot 0.2504; 6.8832, 0.5\cdot 0.0545; 10.5144, 0.5\cdot 0.2807;$$

$$-7.251, 0.5\cdot 0.0222; -0.732, 0.5\cdot 0.0286; 5.7871, 0.5\cdot 0.0583; 12.3063, 0.5\cdot 0.0842; 18.825, 0.5\cdot 0.1266; 0, 0.5\cdot 0.6801].$$

We have $P^{AS}(\mathbf{g}'') > 1.24$.