Contracts are not Salaries in the Hidden-Substitutes Domain

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Abstract

We show that many-to-one matching markets with contracts where colleges' preferences satisfy the hidden substitutes condition of Hatfield and Kominers (2015) may not be embedded, in the sense of Echenique (2012) into a Kelso and Crawford (1982) matching-with-salaries market. Our proof relies on a configurations of preferences that is observed in many college admissions markets.

Generalizing the insights of early studies of two-sided matching markets has been a central theme in the matching literature for more than three decades. A seminal contribution by Kelso and Crawford (1982) studies a matching-with-salaries model and finds parallel results to those of Gale and Shapley

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(1962), including a salary-adjustment process resembling the Deferred Acceptance (DA) algorithm. The groundbreaking work of Hatfield and Milgrom (2005) introduces the matching-with-contracts language, and extends the results further to matching-with-contracts markets where hospitals' preferences satisfy the substitutability condition. But a recent influential paper by Echenique (2012) shows that when hospitals' preferences satisfy the substitutes condition of Hatfield and Milgrom (2005), there exists an embedding that injectively maps markets with contracts to Kelso and Crawford (1982) markets with salaries and gross substitutes demands, such that the set of stable allocations is preserved, and if a matching is preferred by some agent in the matching-with-contracts market then the corresponding matching is preferred by the corresponding agent in the matching-with-salaries market.[1] Schlegel (2015) and Jagadeesan (2016) both provide different constructions that embed matching-with-contracts markets into the matching-with-salaries environment, even when hospitals' preferences only satisfy the weaker condition of unilateral substitutability (Hatfield and Kojima, 2010).[2]

This paper focuses on the even more general environments, where hospitals' preferences only satisfy the condition of hidden substitutability (Hatfield and Kominers, 2015), which is a weaker restriction on hospitals' preferences than the unilateral substitutability condition (Kadam, 2017; Zhang, 2016). We establish, via an example, that when hospitals' preferences satisfy only this weaker condition, matching-with-contracts markets may not be embedded into the matching-with-salaries model in the sense of Echenique (2012). Hospitals' preferences in our example also satisfy the bilateral substitutability condition (Hatfield and Kojima, 2010), which is independent of the hidden substitutability condition (Kadam, 2017). Thus, this paper also provides an alternative proof to Echenique's (2012) finding that an embedding may not exist under the bilateral substitutability condition.[3]

The example we describe is not contrived nor is it very complicated. Instead, such preferences are naturally occurring in college admissions markets, and were observed, for example, in the Israeli Psychology Master's Match (Hassidim, Romm and Shorrer, 2016) and in Hungarian college admissions (Biró, 2012; Shorrer and Sóvágó, 2017). In these markets, students may be

[1]Kominers (2012) extends this result to the many-to-many setting.

[2]Sönmez and Switzer (2013) present the cadet-branch matching, which is a real-life application of the matching-with-contracts model which uses the unilateral substitutes condition.

[3]Our example also satisfies the law of aggregate demand (Hatfield and Milgrom, 2005).

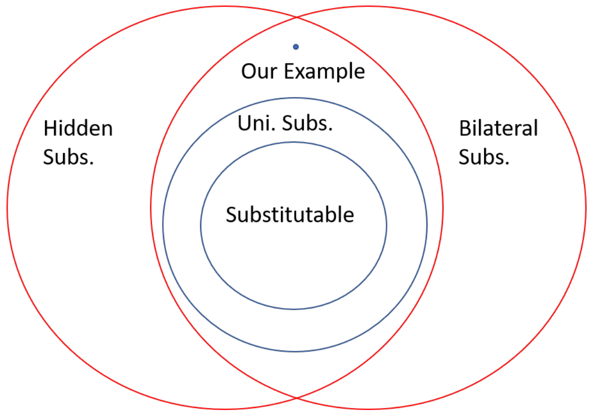

Figure 1: Illustration of the Structure of the Paper. The Venn diagram illustrates the relationship between different restrictions on hospital preferences (Kadam, 2017; Zhang, 2016), and indicates that our non-embeddable example simultaneously satisfies the hidden-substitutes condition and the bilateral substitutes condition.

matched to a program under multiple terms (e.g., with our without a scholarship) and programs wish to admit the best students under multiple capacity constraints (e.g., on the number of available seats and on the number of available scholarships).[4] In this respect, our second contribution is in identifying several real-life markets that fall under the matching-with-contracts framework, but cannot be embedded into a Kelso and Crawford (1982) labor market model.

1 Notation and Definitions

Following the notation of Hatfield and Milgrom (2005), let D be a finite set of *doctors*, H a finite set of *hospitals*, and T a finite set representing potential *contractual terms* between doctors and hospitals. A *contract* is a tuple $(d, h, t) \in D \times H \times T$ that specifies a doctor, a hospital, and their contractual terms. Denote the set of all possible contracts by $X \subset D \times H \times T$.

[4]Hassidim, Romm and Shorrer (2017) present other interesting consequences of this preference structure, including asymptotic manipulability of DA and variance in class sizes across stable matchings.

Denote by X_i the subset of contracts in X that involve agent i. For each $Y \subset X$, let $Y_i := Y \cap X_i$. A subset Y of X is *feasible* if for all $d \in D$ we have $|Y_d| \leq 1$. The feasibility restriction reflects our focus on many-to-one matching markets, where doctors cannot work for more than one hospital.

Each agent, i, has strict preferences over feasible subsets of X_i, denoted by $\succ_i$. Agent i's preferences induce a *choice function*, $\mathrm{Ch}_i : 2^X \to 2^{X_i}$, that identifies the subset of Y_i most preferred by i for any subset Y of X.[5]

An allocation is a subset Y of X. An allocation Y is *individually rational* if, for each i, $\mathrm{Ch}_i(Y_i) = Y_i$. The allocation Y is *unblocked* if there does not exist a hospital, h, and a non-empty $Z \subset X_h \setminus Y$ with $Z_i \subset \mathrm{Ch}_i(Z_i \cup Y_i)$ for all $i \in H \cup D$. An allocation Y is *stable* if it is both individually rational and unblocked.

Definition 1 (Hatfield and Milgrom (2005))**.** $\mathrm{Ch}_h(\cdot)$ satisfies *substitutability* if $Z_h \subset Y_h$ implies that $Z_h \setminus \mathrm{Ch}_h(Z_h) \subset Y_h \setminus \mathrm{Ch}_h(Y_h)$, for any Z and Y.

Definition 2 (Hatfield and Kojima (2010))**.** $\mathrm{Ch}_h(\cdot)$ satisfies *unilateral substitutability* if $X_d \cap Z_h = \emptyset$ implies that $Z_h \setminus \mathrm{Ch}_h(Z_h) \subset (Z_h \cup \{x\}) \setminus \mathrm{Ch}_h(Z_h \cup \{x\})$ for any Z, any $d \in D$, and any $x \in X_d$

Definition 3 (Hatfield and Kominers (2015))**.** $\mathrm{Ch}_h(\cdot)$ satisfies *hidden substitutability* if there exists an order over 2^{X_h}, $\bar{\succ}_h$, such that $\bar{\succ}_h$ and $\succ_h$ agree on all pairs of feasible subsets of X_h, and $\bar{\succ}_h$ induces a substitutable choice function (which may choose infeasible allocations).

Definition 4 (Hatfield and Kojima (2010))**.** $\mathrm{Ch}_h(\cdot)$ satisfies *bilateral substitutability* if for any Z, $d, d' \in D$ and any $x \in X_d$, $y \in X_{d'}$, we have that $X_d \cap Z_h = X_{d'} \cap Z_h = \emptyset$ and $x \in \mathrm{Ch}_h(Z_h \cup \{x, y\})$ imply $x \in \mathrm{Ch}_h(Z_h \cup \{x\})$.

2 Main Result

Lemma 1. *A matching-with-contracts market with no doctor-optimal stable matching cannot be embedded in a Kelso and Crawford labor market.*

The Lemma follows from the definition of an embedding and the fact that a worker-optimal stable matching is guaranteed to exist in matching-with-salaries markets (Kelso and Crawford, 1982).

[5]Since preferences are strict, $\mathrm{Ch}_i(\cdot)$ is indeed a function, and it satisfies the irrelevance of rejected contracts condition (Aygün and Sönmez, 2013).

Proposition 1. *Matching-with-contracts markets where hospitals' preferences satisfy both the bilateral substitutes condition and the hidden substitutes condition may not be embedded (in the sense of Echenique, 2012) in Kelso–Crawford matching-with-salaries markets with gross substitutes demands.*

Proof. Consider a market with $D = \{d_1, d_2\}$, $H = \{h\}$, and $T = \{m, a\}$, where m corresponds to the morning shift, and a corresponds to the afternoon shift. Hospital h's preferences over acceptable,[6] feasible allocations are:

$$h : \{(d_1, h, m), (d_2, h, a)\} \succ_h \{(d_1, h, a), (d_2, h, m)\}$$
$$\succ_h \{(d_1, h, m)\} \succ_h \{(d_1, h, a)\} \succ_h \{(d_2, h, m)\} \succ_h \{(d_2, h, a)\} \succ_h \emptyset.$$

These preferences reflect the hospital's desire to man both shifts, as well as its stronger preference for doctor d_1 and for providing high quality care in the morning shift. Doctors' preferences are:

$$d_1 : (d_1, h, m) \succ_{d_1} (d_1, h, a) \succ_{d_1} \emptyset$$
$$d_2 : (d_2, h, m) \succ_{d_2} \emptyset$$

Namely, both doctors prefer the morning shift to the afternoon shift, and d_2 prefers her outside option to the afternoon shift.

The preferences of the hospital h violate the unilateral substitutability condition, and hence the more restrictive substitutability condition. To see this, note that

$$(d_1, h, m) \in \mathrm{Ch}_h (\{(d_1, h, m), (d_2, h, m)\}) = \{(d_1, h, m)\},$$

but

$$(d_1, h, m) \notin \mathrm{Ch}_h (\{(d_1, h, m), (d_2, h, m), (d_1, h, a)\}) = \{(d_1, h, a), (d_2, h, m)\}.$$

Hospital h's preferences clearly satisfy the bilateral substitutes condition. To see that they also satisfy the hidden substitutability condition, consider the following completion of $\succ_h$:

$$\{(d_1, h, a), (d_1, h, m)\} \bar{\succ}_h \{(d_1, h, m), (d_2, h, a)\} \bar{\succ}_h \{(d_1, h, a), (d_2, h, m)\}$$
$$\bar{\succ}_h \{(d_1, h, m)\} \bar{\succ}_h \{(d_1, h, a)\} \bar{\succ}_h \{(d_2, h, m)\} \bar{\succ}_h \{(d_2, h, a\} \bar{\succ}_h \emptyset.$$

[6] We follow the convention of omitting unacceptable allocations from the description of preferences.

The matching-with-contracts market described above has two stable allocations, $\{(d_1, h, m)\}$, which is the result of the doctor-proposing DA, and $\{(d_1, h, a), (d_2, h, m)\}$. Notably, the outcome of DA is not the stable allocation most preferred by both doctors. In fact, a doctor-optimal stable allocation does not exist. To see this, note that the only allocation that both doctors weakly prefer to the above-mentioned stable allocations has both of them work in the morning ($t = m$), but this allocation is not acceptable to the hospital. We also note that the two allocations have different numbers of assigned doctors.

The proof follows from Lemma 1 by the inexistence of a doctor-optimal stable matching in the matching-with-contracts market. □

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