# Justified-Envy-Minimal Efficient Mechanisms for Priority-Based Matching*†

Hyukjun Kwon
Rutgers University

Ran I. Shorrer
Penn State

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**Abstract**

We study priority-based matching markets with public and private endowments. We propose a novel partial order for comparing matching mechanisms in terms of their "fairness." Using this order, we show that efficiency-adjusted deferred acceptance (EADA) is justified-envy minimal in the class of efficient mechanisms, while top trading cycles (TTC) and other popular mechanisms are not. Our findings highlight EADA as an interesting alternative to TTC in the context of transplantation-organ markets. Restricting attention to strategyproof mechanisms, we show that TTC is justified-envy minimal, providing robustness to the result of Abdulkadiroğlu et al. (2020).

## 1 Introduction

Efficiency and fairness are two key desiderata in social choice. We study priority-based matching, where these desiderata are, unfortunately, incompatible. Many important indivisible resources are allocated based on agents' preferences and priorities, including school seats, public housing, and organs for transplantation. Priorities typically represent the precedence of agents' claim to each good which reflect a public policy choice. Violations of priorities are, thus, a deviation from the stated precedence order, which may be considered unethical or unfair, potentially justifying complaints or lawsuits.[1]

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[1] See Ehlers and Morrill (2019) and references therein.

Motivated by school choice, Abdulkadiroğlu and Sönmez (2003) study priority-based matching with public endowment. They show that each desideratum can be achieved by a strategyproof mechanism—another important desideratum when agents' preferences are unknown.[2] Furthermore, they point to two strategyproof mechanisms: the efficient top trading cycles (TTC) mechanism, and the fair deferred acceptance (DA) mechanism. Kesten (2010) suggests a third mechanism, the efficiency adjusted deferred acceptance (EADA) mechanism. These mechanisms became focal in the large literature that followed.

Balinski and Sönmez (1999) show that, like all other fair mechanisms, the matching achieved by DA is not necessarily Pareto efficient. However, they show that it is constrained efficient in the sense that any matching that Pareto dominates it is not fair. While, logically, this statement defines a set, it turns out that the outcome of DA is the only member of this set.[3] In other words, the Pareto-frontier of fair matchings consists only of the outcome of DA. More simply, the outcome of DA Pareto dominates any fair matching.

In this paper, our goal is to address the dual question. We propose two partial "fairness" orders over mechanisms (described below). This allows us, given a collection of mechanisms, to ask which ones are closest to being fair? That is, which ones are justified-envy minimal? We are especially interested in the collection of Pareto-efficient mechanisms. Considering this class of mechanisms is appropriate in environments where agents cannot misrepresent their preferences. This is the case, for example, when preferences depend on observable characteristics (as is often the case for when medical resources, such as organs for transplantation, are allocated).[4]

[2]We refer to a mechanism as strategyproof if agents have a weakly dominant strategy of reporting their preferences truthfully.

[3]This may not be the case when objects' priorities are not strict (Erdil and Ergin, 2008).

[4]For example, in the context of liver exchange, Ergin et al. (2018) assume that preferences over potential grafts are public information *"[s]ince it purely depends on observable donor characteristics and determined based on agreed-upon medical criteria ... "* Similarly, in the context of kidney exchange, Ashlagi and Roth (2014) report that *"during the initial startup period, attention to the incentives of patients and their surgeons to reveal information was important. But as infrastructure has developed, the information contained in blood tests has come to be conducted and reported in a more standard manner (sometimes at a centralized testing facility), reducing some of the choice about what information to report and with what accuracy. So some strategic issues have become less important over time (and indeed current practice does not deal with the provision of information that derives from blood tests as an incentive issue)."*

There is some evidence that transplantation centers engage in strategic behavior, e.g. by conducting easily arranged exchanges internally (Ashlagi and Roth, 2014) or choosing treatment

In addition to the full class of Pareto-efficient mechanisms, we also consider the subclass of strategyproof Pareto-efficient mechanisms. Focusing on this subclass is appropriate in environments like centralized school-choice where strategic agents report their preferences. In both cases we allow for both public and *private* endowments. This generality is crucial for applications such as organ exchange (Roth et al., 2004), house allocation (Abdulkadiroğlu and Sönmez, 1999), and teacher assignment subject to tenure (Pereyra, 2013; Combe et al., 2018; Gonczarowski et al., 2019).

Market clearing algorithms that respects private endowments may prove useful in markets where a centralized marketplace does not exist yet, but where there is a potential for such marketplace to be beneficial. For example, the human-plasma market has a similar market structure to that of the market for transplantation kidneys.[5] Plasma has a variety of medical uses, among them is the use of recovered patients' plasma for post-exposure prophylaxis and the treatment of several infectious diseases (examples include SARS-CoV, MERS, and Ebola. See Bloch et al., 2020).

We propose two partial "fairness" orders over mechanisms. The first partial order is novel to this work and compares the set of justified-envy triplets (with respect to set inclusion). We say that $(i, j, s)$ is a justified-envy triplet when student $i$ has justified envy towards student $j$ at school $s$ (we provide a formal treatment in Section 2). The second partial order compares jusitified-envy pairs; it has been used in previous papers (e.g. Abdulkadiroğlu et al., 2020). We say that $(i, s)$ is a justified-envy pair if there exists a student $j$ such that $(i, j, s)$ is a justified-envy triplet.

Using both partial orders, we show that (1) EADA is maximally fair (justified-envy minimal) in the class of Pareto-efficient mechanisms. And, (2) in one-to-one matching markets, TTC is maximally fair (justified-envy minimal) in the class of

protocols that improve their patients' priority (Varshney et al., 2020). Our notion of strategyproofness does not consider such behaviors. It may be possible to address these issues separately using the fact that transplantation centers are repeat players (Ashlagi and Roth, 2014; Liu, 2020).

[5]Since human plasma contains antibodies against other blood types, in this market the lattice of blood-type compatibility is flipped, with O-patients being easy to match and AB-patients being the hardest. Here, too, there are consideration regarding other aspects including the presence of HLA antibodies (see Bloch et al., 2020, and references therein). We are grateful to Assaf Romm and Tayfun Sönmez for drawing our attention to this market.

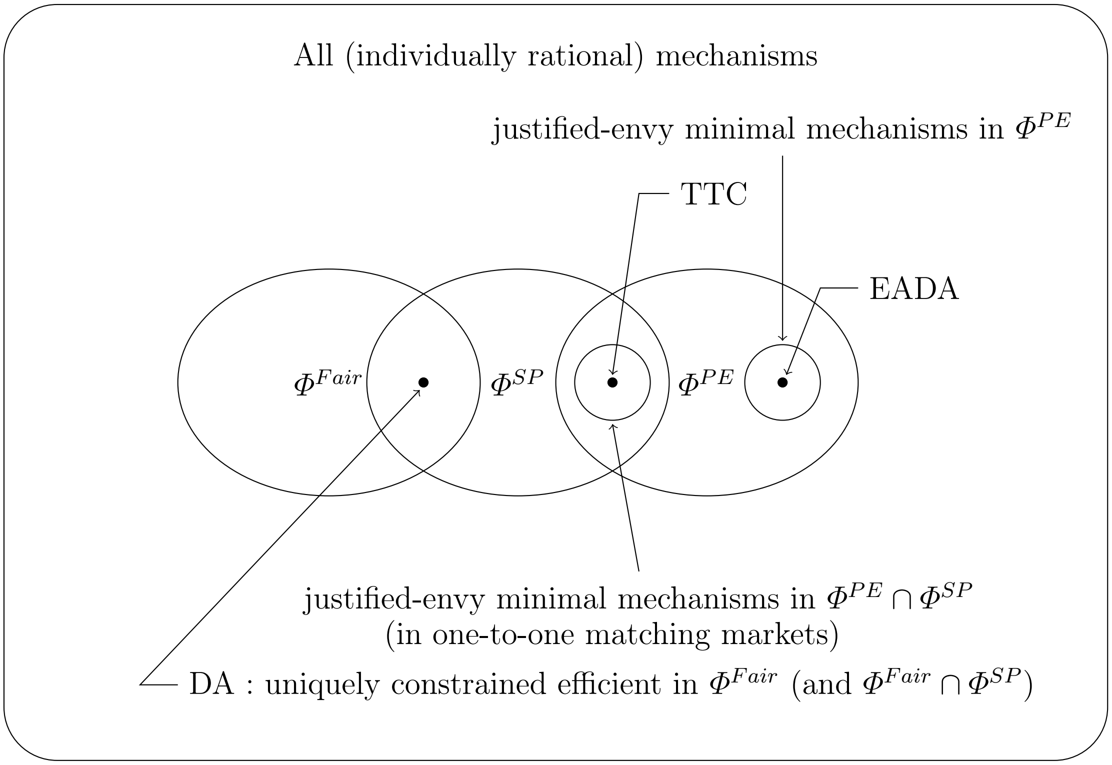


Figure 1: The figure summarizes our main results. The rectangle represents the space of all mechanisms (or all individually rational mechanisms). $\Phi^{Fair}$, $\Phi^{SP}$, and $\Phi^{PE}$ are the collection of fair, strategyproof, and Pareto-efficient mechanisms, respectively. Justified-envy minimal mechanisms in $\Phi^{PE}$ are not strategyproof (Proposition 3), and include EADA (Theorem 2). However, this collection is not a singleton (Proposition 2). TTC is justified-envy minimal in $\Phi^{PE} \cap \Phi^{SP}$ in one-to-one matching markets (Theorem 3), but may not be the unique mechanism with this property (Abdulkadiroğlu et al., 2020). The rest of the relations follow from Abdulkadiroğlu and Sönmez (2003).

Pareto-efficient and strategyproof mechanisms.[6] We also show that TTC, and all other strategyproof mechanisms, is not justified-envy minimal in the unrestricted class of Pareto-efficient mechanisms. Both EADA and TTC are individually rational, which is crucial in the presence of private endowments. Figure 1 summarizes these results.

While EADA is justified-envy minimal in the class of Pareto-efficient mechanisms with respect to both orders, other popular Pareto-efficient mechanisms such as the immediate acceptance (Boston) mechanism, serial dictatorship (SD), TTC,

[6]This second result has been established by Abdulkadiroğlu et al. (2020) with respect to justified-envy pairs for the case of public endowment.

DA followed by TTC (DA+TTC), and variations of TTC, such as Clinch and Trade (Morrill, 2015), and Equitable TTC (Hakimov and Kesten, 2018) are not justified-envy minimal in this class with respect to neither order. Our finding, thus, formalizes the implicit intuition in Morrill (2015) that EADA is more fair relative to other Pareto-efficient mechanisms. Morrill (2015) suggests that if a policymaker *"values efficiency first, fairness second, and strategyproofness third, then she should run the efficiency-adjusted deferred acceptance algorithm."*[7]

Combined with our negative results on TTC, our findings present EADA as an interesting alternative to TTC in contexts such as transplantation-organ markets: insofar as priorities reflect the policymaker's goals,[8] EADA results in allocations that are maximally consistent with these goals in the class of Pareto-efficient mechanisms.

We are not aware of previous papers that suggested using EADA in the presence of private endowments. Roth et al. (2004) were the first to propose the use of TTC for organ exchanges. Due to the prominence of this application, and since we have shown that EADA is justified-envy minimal among Pareto-efficient mechanisms while TTC is not, in Section 7 we compare the two mechanisms in another important dimension: the number of transplantations that they generate. We find that the mechanisms are not comparable in this dimension: EADA can generate more or fewer transplantations than TTC, depending on the market conditions.[9]

Our paper directly relates to Abdulkadiroğlu et al. (2020). They shows that in one-to-one markets with public endowment TTC is justified-envy minimal in the class of strategyproof and Pareto-efficient mechanisms with respect to justified-envy pairs,[10] and provide supporting empirical evidence from the school-choice context. We provide robustness to their results by showing that they also hold

[7]Other justifications for EADA are provided in Doğan and Yenmez (2017), Tang and Zhang (2020), Troyan et al. (2020), Ehlers and Morrill (2019), Dur et al. (2019) and Doğan and Ehlers (2020b).

[8]In transplantation-organ markets in the Netherlands, organ-recipients with the smallest chance of finding another compatible donor in the pool are ranked higher (Keizer et al., 2005). In the United States, the priority for deceased-donor organs is based on a points system with certain organs using different formulae (Israni et al., 2014).

[9]Afacan and Dur (forthcoming) and Afacan et al. (2020) study assignemnt maximization in different settings.

[10]This partial order is also used in Tang and Zhang (2020) to derive the notion of weak stability, self-constrained efficiency, and self-constrained optimality, and in Combe et al. (2018) to compare teacher assignments.

with respect to justified-envy triplets and in the presence of private endowments.[11] Our findings contribute an explanation as to why this mechanism became focal in the literature.[12]

The recent study of Doğan and Ehlers (2020b) considers a wide array of orders and shows that EADA is justified-envy minimal with respect to some (including the partial order based on pairs), but not with respect to others. Abdulkadiroğlu and Grigoryan (2020) focus on identifying justified-envy-minimal Pareto-efficient allocations in the presence of coarse priorities.

This paper is related to the large and growing literature on matching theory and its applications in market design (see Roth, 2018). While we believe our ideas may prove useful in many real-life market (especially the idea of using EADA with private endowments), we wish to highlight that we study the properties of certain mechanisms, but not whether they are appropriate for any particular market. We leave for future studies the challenge of assessing the fit of these mechanisms as part of a comprehensive market design solution in specific markets.

# 2 Model

## 2.1 Definitions and Notation

An *allocation problem* is a sextuple $\mathcal{P} \equiv (I, S, q, o, P, \succ)$, where $I = \{i_1, i_2, \ldots, i_n\}$ is a finite set of agents, $S = \{s_1, s_2, \ldots, s_m\}$ is a finite set of objects, $q = (q_s)_{s\in S}$ is a vector of object capacities with $q_s \in \mathbb{N}$ for all $s \in S$.

$P = (P_i)_{i\in I}$ is a profile of strict agent preferences over objects and being unassigned (i.e., each $P_i$ is a complete, transitive, and irreflexive relation over $S \cup \{i\}$, where $i$ represents being unassigned), and $\succ = (\succ_s)_{s\in S}$ is a vector of strict object priorities (complete, transitive, and irreflexive relations over agents). Denote by $R_i$ the at-least-as-good-as relation associated with $P_i$.

Each object may be part of the *private endowment* of a single agent $i \in I$ (we assume that privately owned objects are distinct). If an object does not belong to any private endowment, we say that it is in the *public endowment* or *publicly owned*. Formally, an *ownership structure* $o$ is a function from $S$ to $I \cup \{\emptyset\}$ where $\emptyset$

[11] The contemporaneous study of Doğan and Ehlers (2020a) provides a general result that nest the result of Abdulkadiroğlu et al. (2020) as well as ours.

[12] Other justifications for TTC are provided in Ma (1994); Abdulkadiroğlu and Sönmez (1999); Pápai (2000); Chen and Sönmez (2002); Pycia and Ünver (2017); Dur (2012); and Morrill (2015).

represents public ownership. The priorities of privately owned objects must respect the ownership structure—i.e., they are restricted to rank the owner first—but the policymaker is free to pre-specify the rest of the priorities. The (pure) public endowment case corresponds to the special case of our model where $o(\cdot) \equiv \emptyset$.

A *matching* $\mu$ for given allocation problem $\mathcal{P}$ is a function from $I$ to $S \cup I$ such that 1) $|\mu^{-1}(s)| \leq q_s$ for each $s \in S$, and 2) $\mu(i) \notin S \Rightarrow \mu(i) = i$ for each $i \in I$. A *mechanism* is a function that maps each allocation problem to a matching for this allocation problem.

Given an allocation problem, matching $\mu$ *Pareto dominates* matching $\mu'$ if $\mu(i) R_i \mu'(i)$ for all $i \in I$ and $\mu(i) P_i \mu'(i)$ for some $i \in I$. Equivalently, for any profile, $u$, of utility functions that agree with $P$ (and any profile of positive Pareto weights, $\lambda$), $\sum_i \lambda_i u_i(\mu(i)) > \sum_i \lambda_i u_i(\mu'(i))$. A matching is *Pareto efficient* if no matching Pareto dominates it. A mechanism is *Pareto efficient* if it selects a Pareto-efficient matching in every allocation problem. The mechanism $\varphi$ Pareto dominates $\varphi'$ if $\varphi(\mathcal{P})(i) R_i \varphi'(\mathcal{P})(i)$ for all $\mathcal{P}$ and $i \in I$, and $\varphi(\mathcal{P})(i) P_i \varphi'(\mathcal{P})(i)$ for some $\mathcal{P}$ and $i \in I$. Given a collection of mechanisms, $\Phi$, we say that the mechanism $\varphi \in \Phi$ is *constrained efficient in* $\Phi$ if there is no mechanism $\varphi' \in \Phi$ that Pareto dominates $\varphi$.

Given matching $\mu$, we say agent $i$ has *justified envy towards agent* $j$ *at object* $s$ if $\mu(j) = s$, $s P_i \mu(i)$ and $i \succ_s j$.[13] We call the triplet $(i, j, s)$ a *t-justified envy* and the pair $(i, s)$ a *p-justified envy* (of note, p-justified envy ignores the quantity and the identities of the envied agents). A matching is *fair* if it induces no t-justified envy (and thus no p-justified envy). A mechanism is *fair* if it selects a fair matching for all allocation problems.

A matching $\mu$ is *individually rational* if $\mu(i) R_i i$ and $\mu(i) R_i s$ for all $s \in o^{-1}(i)$, for all $i \in I$. A mechanism is *individually rational* if it selects an individually rational matching in every allocation problem.

Lastly, a mechanism $\varphi$ is *strategyproof* if agents cannot benefit from misrepresenting their preferences. Formally, for each $\mathcal{P} = (I, S, q, o, P, \succ)$, for each agent $i \in I$, and any $\mathcal{P}' = (I, S, q, o, P'_i, P_{-i}, \succ)$, we have $\varphi(\mathcal{P})(i) R_i \varphi(\mathcal{P}')(i)$.[14]

[13] For a definition of justified envy in more general domains see Romm et al. (2020).

[14] Our notion of strategy does not allow agents to misrepresent information about their private endowment (e.g., by hiding the existence of objects they own). Cf. Ergin et al. (2018).

### 2.1.1 Discussion: Public vs. Private Endowments

Private ownership introduces some distinct considerations to the problem of object allocation. For example, how to address conflicts between priorities and ownership. Additionally, although efficiency is independent of the ownership structure, individual rationality is more restrictive in the presence of private endowments, and so is fairness (for a comprehensive discussion, see Haeringer (2018), Chapter 11).

We use a simple, but useful and reasonable method that allows us to accommodate the private ownership structure without creating additional complexities. As mentioned above, we make the priority structure reflect the private ownership by requiring privately owned objects to rank their owners first. This method has a number of benefits. First, mechanisms that are introduced in purely public endowments, such as EADA, can be naturally implemented even in the presence of the private ownership. As a consequence, EADA can be used in the transplantation-organ markets (e.g., kidney exchange problems). Second, we can invoke several known results on these mechanisms that were proved only for the case of public endowment. Third, we may maintain the definition of properties, such as fairness, as defined in public endowments. Note that the definition of fairness in the previous section uses only priorities, not ownership. However, since the owner has top priorities, if a mechanism is fair in our definition, it never makes agents envious of someone that occupy their private endowments, essentially respecting ownership.

It is worth noting that, the ownership structure puts a restrictions on $\succ$. Thus, requirements that must hold over all possible priority profiles (such as strategyproofness of a matching algorithm or the fairness comparisons described below) are simpler to satisfy, as fewer conditions must hold.

## 2.2 Justified-Envy Minimality

Given an allocation problem $\mathcal{P}$ and a matching $\mu$, let $JE^{\mu}(\mathcal{P})$ be the set of t-justified envies at $\mu$. When there is no risk of confusion, we suppress the dependence on $\mathcal{P}$. For a fixed allocation problem, the matching $\mu$ has *weakly less t-justified envy* than $\mu'$ if $JE^{\mu} \subseteq JE^{\mu'}$. Furthermore, $\mu$ has *less t-justified envy* than $\mu'$ if $JE^{\mu} \subsetneq JE^{\mu'}$. Given an allocation problem and a set of matchings, $\mathcal{M}$, we say that the matching $\mu \in \mathcal{M}$ is *t-justified-envy minimal in* $\mathcal{M}$ if there is no

$\mu' \in \mathcal{M}$ with less t-justified envy than $\mu$.

The mechanism $\varphi$ has *weakly less t-justified envy* than mechanism $\varphi'$ if for all $\mathcal{P}$, $JE^{\varphi(\mathcal{P})}(\mathcal{P}) \subseteq JE^{\varphi'(\mathcal{P})}(\mathcal{P})$. The mechanism $\varphi$ has *less t-justified envy* than mechanism $\varphi'$ if $JE^{\varphi(\mathcal{P})}(\mathcal{P}) \subseteq JE^{\varphi'(\mathcal{P})}(\mathcal{P})$ for all $\mathcal{P}$, and $JE^{\varphi(\mathcal{P})}(\mathcal{P}) \subsetneq JE^{\varphi'(\mathcal{P})}(\mathcal{P})$ for some $\mathcal{P}$. Given a collection of mechanisms, $\Phi$, we say that the mechanism $\varphi \in \Phi$ is *t-justified-envy minimal in $\Phi$* if there is no $\varphi' \in \Phi$ which has less t-justified envy than $\varphi$. The definitions for p-justified envy are analogous, using the sets of p-justified envies, $\widehat{JE}^{\mu}(\mathcal{P})$.

### 2.2.1 Interpretation

The partial orders underlying our definitions of p- and t-justified-envy minimality are somewhat coarse. There are many other reasonable, finer criteria. One broad class is captured by justified-envy aggregators. A *t-justified-envy aggregator* is given by a vector of positive numbers $\lambda = \left(\lambda_{(i,j,s)}\right)_{(i,j,s)\in I\times I\times S}$. We say that the matching $\mu$ has less $\lambda$-t-justified envy than the matching $\mu'$ if $\sum_{(i,j,s)\in JE^{\mu}} \lambda_{(i,j,s)} < \sum_{(i,j,s)\in JE^{\mu'}} \lambda_{(i,j,s)}$. The definitions for p-justified envy are analogous. Specific aggregators may be more appropriate in particular applications, depending on the setting. For example, it may be appropriate to assign higher weights to pairs or triplets that involve a member of an underrepresented minority or ones that involve highly sought-after schools.

Theorem 1 establishes that a matching is (p-) t-justified envy minimal if and only if it is minimal with respect to some (p-) t-justified-envy aggregator. It implies that the partial orders "less t-justified envy" and "less p-justified envy" coincide with the intersection of the orderings induced by all of the respective justified-envy aggregators (Corollary 1). This relation resembles the relation between Pareto efficiency and the maximization of weighted utility aggregators or the relation between expected utility maximization and first order stochastic dominance. Thus, in the same fashion, our results can be interpreted as a robust recommendation.

**Theorem 1.** *For any allocation problem and a set of matchings, $\mathcal{M}$, $\mu \in \mathcal{M}$ is (p-) t-justified-envy minimal if and only if there exists a (p-) t-justified-envy aggregator such that $\mu$ is $\lambda$-t-justified-envy ($\lambda$-p-justified-envy) minimal.*

*Proof.* Assume that $\mu$ is not t-justified-envy minimal in $\mathcal{M}$. Then, there exists $\mu' \in \mathcal{M}$ such that $JE^{\mu'} \subsetneq JE^{\mu}$, and so $\sum_{(i,j,s)\in JE^{\mu'}} \lambda_{(i,j,s)} < \sum_{(i,j,s)\in JE^{\mu}} \lambda_{(i,j,s)}$

since the right-hand side summation is over a superset of elements and all entries of $\lambda$ are positive. Thus, $\mu$ is not $\lambda$-t-justified-envy minimal in $\mathcal{M}$.

Conversely, let $\mu$ be t-justified-envy minimal in $\mathcal{M}$. Then, for any $\mu' \in \mathcal{M}$, we have either $JE^{\mu'} = JE^{\mu}$ or $JE^{\mu'} \setminus JE^{\mu} \neq \emptyset$. Let $\varepsilon > 0$ be a sufficiently small number, and consider the t-justified-envy aggregator given by the vector $\lambda$ such that $\lambda_{(i,j,s)} = \varepsilon$ for each $(i, j, s) \in JE^{\mu}$ and $\lambda_{(i,j,s)} = 1/\varepsilon$ otherwise. This aggregator assigns to $\mu$ a weakly lower value than to any other $\mu' \in \mathcal{M}$ (strictly lower if $JE^{\mu'} \neq JE^{\mu}$). The proof for p-justified envy is completely analogous. □

**Corollary 1.** *For any pair of matchings $\mu$ and $\mu'$:*

1. $JE^{\mu} \subsetneq JE^{\mu'}$ *if and only if $\mu$ has less $\lambda$-t-justified envy than $\mu'$ for any t-justified-envy aggregator (given by $\lambda$).*
2. $\widehat{JE}^{\mu} \subsetneq \widehat{JE}^{\mu'}$ *if and only if $\mu$ has less $\lambda$-p-justified envy than $\mu'$ for any p-justified-envy aggregator (given by $\lambda$).*

Our approach presents a direction for systematically extending the partial orders we use (which rely on set inclusion). For example, in specific contexts, one may be willing to assume that if $1 \succ_s 2 \succ_s 3$, then $\lambda_{(1,2,s)} < \lambda_{(1,3,s)}$. The partial order resulting from the intersection of all aggregators that satisfy this restriction would deem $\{(1, 3, s)\}$ as having more justified envy than $\{(1, 2, s)\}$. Continuing the analogy to expected utility maximization, an additional assumption in that domain can be that utility functions are concave, and the resulting extension is second order stochastic dominance.

### 2.2.2 Independence of t- and p-Justified-Envy Minimality

**Proposition 1.** *A matching can be t-justified-envy minimal and not p-justified-envy minimal in a set of matchings, and vice versa.*

*Proof.* Consider the following allocation problem: $I = \{1, 2, 3, 4\}$, $S = \{a, b, c\}$, $q_a = 2, q_b = 1, q_c = 1$, all objects are public ($o(\cdot) \equiv \emptyset$), and $(P, \succ)$ as follows:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $c$ | 4 | 1 | 1 |
| $b$ | $b$ | $b$ | $a$ | 1 | 2 | 2 |
| $c$ | $c$ | $c$ | $\vdots$ | 2 | 3 | 3 |
| 1 | 2 | 3 | | 3 | 4 | 4 |

Let $\{\mu^1, \mu^2, \mu^3\}$ be a set of matchings as follows:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu^1$ | $b$ | $a$ | $a$ | $c$ |
| $\mu^2$ | $b$ | $a$ | $c$ | $a$ |
| $\mu^3$ | $c$ | $a$ | $b$ | $a$ |

We have that $JE^{\mu^1} = \{(1,2,a),(1,3,a)\}$, $JE^{\mu^2} = \{(1,2,a)\}$, and $JE^{\mu^3} = \{(1,2,a),(1,3,b)\}$. Hence, $\widehat{JE}^{\mu^1} = \widehat{JE}^{\mu^2} = \{(1,a)\}$ and $\widehat{JE}^{\mu^3} = \{(1,a),(1,b)\}$. Therefore, in the set $\{\mu^1, \mu^2\}$, both matchings are p-justified-envy minimal, while only $\mu^2$ is t-justified-envy minimal. Similarly, in the set $\{\mu^1, \mu^3\}$, both matchings are t-justified-envy minimal, while only $\mu^1$ is p-justified-envy minimal. □

**Corollary 2.** *The partial orders "less t-justified envy" and "less p-justified envy" are independent.*

In allocation problems where each object has capacity of one, the order based on pairs extends the order based on triplets (i.e., makes more comparisons). Building on the interpretation presented in the previous section, the partial order based on pairs can be interpreted as imposing t-justified-envy aggregators the restriction that $\lambda_{(i,j,s)} = \lambda_{(i,k,s)}$ for any $i, j, k \in I$ and any $s \in S$ (this is possible since, in this class of allocation problems, only one person can be envied in each school).

**Lemma 1.** *In allocation problems where each object has capacity of one, for any set of matchings $\mathcal{M}$, if $\mu$ is p-justified-envy minimal in $\mathcal{M}$, then it is t-justified-envy minimal in $\mathcal{M}$.*

*Proof.* Toward contradiction, suppose that $\mu \in \mathcal{M}$ is p-justified-envy minimal in $\mathcal{M}$ but not t-justified-envy minimal. Then there exists $\mu' \in \mathcal{M}$ with less t-justified envy than $\mu$.

We note that justified-envy pairs correspond to the first and third coordinates of justified-envy triplets. Furthermore, since objects' capacities are one, each justified-envy pair appears in the set of justified-envy triplets only once. Therefore, a matching that induces (weakly) less t-justified envy also induces (weakly) less p-justified envy. This implies that $\mu'$ has less p-justified envy than $\mu$ contradicting p-justified-envy minimality of $\mu$. □

The following example shows that the converse of Lemma 1 does not hold.

**Example.** *Consider the following allocation problem:* $I = \{1, 2, 3, 4\}$*,* $S = \{a, b, c, d\}$*, and each object has capacity of 1. The table below describes* $(P, \succ)$*.*

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $d$ | 1 | 2 | $\vdots$ | 2 |
| $c$ | $b$ | $a$ | $b$ | 3 | 4 | | 4 |
| $\vdots$ | $d$ | $\vdots$ | $\vdots$ | 2 | 3 | | $\vdots$ |
| | $\vdots$ | | | $\vdots$ | $\vdots$ | | |

*Let* $\mathcal{M} = \{\mu^1, \mu^2\}$*, where* $\mu^1$ *and* $\mu^2$ *are as follows:*

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu^1$ | $c$ | $d$ | $a$ | $b$ |
| $\mu^2$ | $c$ | $a$ | $b$ | $d$ |

*We have* $JE^{\mu^1} = \{(1, 3, a), (2, 4, b)\}$*,* $JE^{\mu^2} = \{(1, 2, a)\}$*,* $\widehat{JE}^{\mu^1} = \{(1, a), (2, b)\}$*, and* $\widehat{JE}^{\mu^2} = \{(1, a)\}$*. Therefore, both matchings are t-justified-envy minimal in* $\mathcal{M}$ *while only* $\mu^2$ *is p-justified-envy minimal in* $\mathcal{M}$*.*

### 2.2.3 Discussion: Comparisons via Pairs and Triplets

While comparisons of p-justified envy appeared in multiple earlier papers, t-justified envy is novel to this study. We believe that relying on justified-envy triplet is useful in many contexts. Consider, for example, a public school district that tries to minimize "scandals." The set of t-justified envies allows to tell apart complaints like "a minority student was not accepted to school X while *seven majority* students with *substantially* lower priority gained admission" from complaints like "a minority student was not accepted to school X while *one* other *minority* student with *slightly* lower priority gained admission." By contrast, p-justified envy comparisons will map both statements to "a minority student was not accepted to school X while one or more other (majority or minority) students with lower (to some extent) priority were accepted." By Corollary 1, in such contexts, where weights are not known, our criterion of t-justified-envy minimality is a robust notion.

Additionally, even in contexts where the researcher want to use p-justified envy comparisons, justified-envy triplets can be used for further refinement. For example, one may require that ties are broken based on the degree of justification of envy (e.g., how far below the envious agent the envied agent is ranked).[15]

## 3 A Fair, Strategyproof, and Constrained-Efficient Mechanism

The DA algorithm was introduced by Gale and Shapley (1962). Abdulkadiroğlu and Sönmez (2003) were the first to suggest it in the context of school-choice (allocation problems with public endowment). DA identifies the matching to which the following process converges: 1) each object $s$ repeatedly rejects all but the best ranked $q_s$ applicants according to $\succ_s$, 2) agents repeatedly apply to the best object that did not reject them, and 3) if agents apply to themselves, they stop applying and remain unassigned.

Balinski and Sönmez (1999) show that the DA matching is not necessarily Pareto efficient, but it Pareto dominates all other fair mechanisms.

**Theorem** (Balinski and Sönmez, 1999)**.** *For any allocation problem, the DA matching Pareto dominates all other fair matchings.*

Let $\Phi^{Fair}$, $\Phi^{IR}$ and $\Phi^{SP}$ be the collection of fair, individually rational and strategyproof mechanisms, respectively. The theorem allows us to say that DA is constrained efficient in the collection of all fair mechanisms. Since DA is justified-envy free and non-wasteful, the restrictions imposed on $\succ$ by the ownership structure guarantee that it is individually rational. Roth (1982) and Dubins and Freedman (1981) show that DA is strategyproof.

**Corollary 3.** *DA is the unique constrained-efficient mechanism in* $\Phi^{Fair}$ *and in* $\Phi^{Fair} \cap \Phi^{IR} \cap \Phi^{SP}$.

## 4 Efficient, Constrained-Justified-Envy-Minimal Mechanisms

Kesten (2010) introduces the EADA algorithm. Tang and Yu (2014) provide an alternative algorithm (SEADA) that yields the same outcome. For brevity, we only describe SEADA. We say that object $s$ is *underdemanded* if it never rejects any agent throughout the DA algorithm. The SEADA algorithm proceeds as follows:

[15]We thank an anonymous referee for suggesting this point.

**Round 0** Run the (round 0) DA algorithm.

**Round k, k $\geq$ 1** For agents assigned to underdemanded objects or left unassigned in round $k-1$ DA algorithm, finalize their assignments as round $k-1$ DA outcome. Remove these agents and underdemanded objects. Rerun (round $k$) DA algorithm for the remaining agents and objects.

The algorithm terminates when all agents are removed from the allocation problem, and the agents are assigned to the objects with which they are removed. We call the mechanism that associates to every allocation problem the outcome of the above algorithm EADA.[16]

**Theorem** (Kesten, 2010)**.** *EADA is Pareto efficient.*

**Remark.** *Kesten (2010) is motivated by school choice and therefore frames his model in the context of public endowment. However, since efficiency is independent of the ownership structure, his theorem applies also in the presence of private endowments.*

Theorem 2 provides a partial answer to the question raised at the beginning. EADA is t-justified-envy and p-justified-envy minimal among all Pareto-efficient mechanisms. In fact, the theorem shows something stronger; for any allocation problem, no efficient matching creates less p- or t-justified envy than the matching resulting from EADA.

**Theorem 2.** *For any allocation problem, the EADA matching is t-justified-envy (p-justified-envy) minimal in the set of Pareto-efficient matchings.*

The proof is in Appendix A. Doğan and Ehlers (2020b) independently showed that the EADA matching is p-justified-envy minimal in the set of Pareto-efficient matchings. In Appendix B we show that, in that setting, the main result of Tang and Zhang (2020) can be used to derive a simple proof that the EADA matching is p-justified-envy minimal in the set of Pareto-efficient matchings (see also Doğan and Ehlers, 2020b).

Since EADA Pareto dominates DA (Tang and Yu, 2014) and DA is individually rational, we have the following:

[16] Both Kesten (2010) and Tang and Yu (2014) present a more general set of mechanisms, as they first specify the set of consenting agents. In this paper, we assume that all agents consent, as otherwise the mechanism is not Pareto efficient.

**Corollary 4.** *The EADA matching is t-justified-envy (p-justified-envy) minimal in the set of individually rational and Pareto-efficient matchings.*

We often highlight the special case of one-to-one problems where each agent $i$ owns a single, distinct object $s_i$, such that $s_i P_i i$. Allocation problems with this structure map to real-life scenarios such as kidney exchange (with no public endowment) if one assumes that patient-donor pairs do not have preference over the identity of the recipient of the donor's organ, except that they prefer that the donor does not donate if the patient does not receive an acceptable organ—one that is ranked higher than the directed donor's organ. For simplicity, we label the subset of these problems as *special kidney-exchange problems*. Proposition 2 shows that EADA is not the unique justified-envy-minimal Pareto-efficient mechanism even in this special case. This stands in contrast with DA being the unique constrained-efficient mechanism among fair mechanisms.

**Proposition 2.** *There exist special kidney-exchange problems where the set of efficient matchings has multiple t- and p-justified-envy minimal matchings.*

*Proof.* Let $I = \{1, 2, 3\}$, $S = \{a, b, c\}$, each object has capacity of 1, and $(P, \succ)$ be as follows:

| $P_1$ | $P_2$ | $P_3$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $b$ | $a$ | $a$ | 1 | 3 | 2 |
| $a$ | $c$ | $b$ | 2 | 1 | 3 |
| $c$ | $b$ | $c$ | 3 | 2 | 1 |
| 1 | 2 | 3 | | | |

The profile of priorities $\succ$ is consistent with the ownership structure in which $a$ is in agent 1's endowment, $c$ is in agent 2's endowment, and $b$ is in agent 3's endowment (but it could have also reflected the policymakers' priorities in the purely public or mixed endowment case). For this allocation problem, EADA yields $\mu^1$. Furthermore, $\mu^1, \mu^2$, and $\mu^3$ are all the Pareto-efficient matchings for this allocation problem.

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^1$ | $b$ | $c$ | $a$ |
| $\mu^2$ | $c$ | $a$ | $b$ |
| $\mu^3$ | $b$ | $a$ | $c$ |

Note that neither of the Pareto-efficient matchings is fair; each of them generates a single t-justified envy, $(2, 3, a)$, $(1, 2, a)$, and $(3, 1, b)$, respectively, and a single p-justified envy, $(2, a), (1, a)$, and $(3, b)$, respectively. □

To see that Proposition 2 implies that EADA is not the unique justified-envy-minimal Pareto-efficient mechanism, consider any mechanism that coincides with EADA except in some special-kidney-exchange problem with multiple t- and p-justified-envy minimal efficient matchings, where it chooses another such matching. Such mechanism is also t- and p-justified-envy minimal in $\Phi^{PE}$.

Given this result, it is natural to ask if there are other t-justified-envy (p-justified-envy) minimal mechanisms in $\Phi^{PE}$ that are also strategyproof. Proposition 3 says that this is impossible. Hence, finding t-justified-envy (p-justified-envy) minimal mechanisms in the collection of Pareto-efficient and strategyproof mechanisms is also of interest. Addressing this question is the focus of the next section.

**Proposition 3.** *Any t-justified-envy (p-justified-envy) minimal mechanism in $\Phi^{PE}$ is not strategyproof, even restricting attention to special kidney-exchange problems.*

The proof for the general case was provided in Proposition 1 of Kesten (2010). In Appendix A we modify his proof to accommodate the restriction to special kidney-exchange problems.

# 5 Efficient, Strategyproof, Constrained-Justified-Envy-Minimal Mechanism

Below, we describe the TTC algorithm for object allocation, which was introduced in the school-choice context by Abdulkadiroğlu and Sönmez (2003).

**Step 0** For each object $s$, assign a counter $c_s$ and initialize it to $c_s = q_s$.

**Step k, k $\geq$ 1** If $c_s = 0$, remove object $s$. If agent $i$'s first-choice among options that were not removed is herself, remove agent $i$, and leave her unassigned. If there are no more agents, terminate. Each remaining agent points to her first-choice among remaining objects. Each remaining object points to its highest-priority remaining agent. A cycle consists of objects and agents $(s_1, i_1, s_2, ..., s_L, i_L)$ where $s_l$ points to $i_l$, $i_l$ points to $s_{l+1(mod\, L)}$. Agents and objects cannot be part of more than one cycle, and there must be at least one cycle. Every agent in a cycle is removed and assigned the object she points to. For each object $s$ in a cycle, set $c_s = c_s - 1$. Proceed to step $k+1$.

We call the mechanism that associates to every allocation problem the outcome of the above algorithm TTC. Theorem 3 states that TTC is t-justified-envy (p-justified-envy) minimal in the class of Pareto-efficient and strategyproof mechanisms in one-to-one matching markets.

**Theorem 3.** *When each object has capacity of one, TTC is t-justified-envy (p-justified-envy) minimal in $\Phi^{PE} \cap \Phi^{SP}$. Furthermore, this holds even when restricting attention to special kidney-exchange problems, or to the set of allocation problems with priorities given by $\succ$.*

*Proof.* The result for p-justified envy is proved by Abdulkadiroğlu et al. (2020). While they consider the public endowment case, their result applies also with general ownership structures under our assumption on priorities. The result for t-justified envy then follows from Lemma 1. □

Since TTC is strategyproof and has the mutual-best property (Dur, 2012; Morrill, 2013), it is individually rational, and so we have the following corollary.

**Corollary 5.** *When each object has capacity of one, TTC is t-justified-envy (p-justified-envy) minimal in $\Phi^{PE} \cap \Phi^{SP} \cap \Phi^{IR}$. Furthermore, this holds even when restricting attention to special kidney-exchange problems, or to the set of allocation problems with priorities given by $\succ$.*

# 6 Negative Results for Other Mechanisms

Morrill (2015) points out that TTC may perform unnecessary trades causing unnecessary justified envy. He suggests three alternative mechanisms: Clinch and Trade (C&T), Always Clinch and Trade (AC&T), and First Clinch and Trade (FC&T). Hakimov and Kesten (2018) propose the Equitable TTC (ETTC) mechanism with the similar motivation. All four mechanisms are Pareto efficient and they all coincide with TTC when objects' capacities are one. Only C&T, FC&T, and ETTC are strategyproof.[17]

[17]See Appendix B for details on the mechanisms discussed in this section.

In addition to DA and TTC, Abdulkadiroğlu and Sönmez (2003) also introduced the Boston (B) mechanism. A popular variant of this mechanism is known as Adaptive Boston (AB). Both mechanisms are Pareto efficient, but they are neither strategyproof nor individually rational (Mennle and Seuken, 2014; Dur, 2019).

Another category of mechanisms are ones that run variants of DA and then Gale's top trading cycles algorithm (Shapley and Scarf, 1974) with the resulting matching as initial endowment. We call the variant that uses DA first, DA+TTC, and the variant that uses object-proposing DA first, CPDA+TTC. These mechanisms are Pareto efficient, but not strategyproof (Alva and Manjunath, 2019).

Abdulkadiroğlu et al. (2020) define the class of priority-adjusted TTC. Mechanisms in this class are Pareto efficient and strategyproof. These mechanisms output the outcome of TTC with respect to agents' preferences and some "artificial" priorities. The case where the artificial priorities coincide with the true priorities corresponds to TTC. The case where all objects share the same artificial priorities corresponds to SD. Of note, SD is not individually rational (Abdulkadiroğlu and Sönmez, 1999).

None of the abovementioned mechanisms is t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$ even when restricting attention to special kidney-exchange problems.

**Proposition 4.** *C&T, AC&T, FC&T, ETTC, DA+TTC, CPDA+TTC, B, AB, and all priority-adjusted TTC are not t-justified-envy (p-justified-envy) minimal in* $\Phi^{PE}$ *even when restricting attention to special kidney-exchange problems.*

According to Theorem 3, in one-to-one markets, TTC is t-justified-envy minimal in the class of Pareto-efficient and strategyproof mechanisms. However, we show that if objects can have larger capacities this result no longer holds. Furthermore, none of the Pareto-efficient and strategyproof mechanisms discussed above are t-justified-envy minimal. Since Proposition 1 shows that our two notions of justified-envy minimality are independent in general, this result provides robustness to the negative result of Abdulkadiroğlu et al. (2020) who show it with respect to p-justified envy.

**Proposition 5.** *Any priority-adjusted TTC, C&T, FC&T, and ETTC are not t-justified-envy (p-justified-envy) minimal in* $\Phi^{PE} \cap \Phi^{SP} \cap \Phi^{IR}$.

C&T, AC&T, FC&T, and ETTC are all motivated by a desire to eliminate unnecessary justified envy that TTC creates, and there are evidence that they achieve this goal with respect to some measures (Morrill, 2015; Hakimov and Kesten, 2018). However, these mechanisms may generate more t- and p-justified envy than TTC.

**Proposition 6.** *C&T, AC&T, FC&T, and ETTC may induce less or more t-justified envy (p-justified envy) than TTC.*

# 7 Discussion

Our findings highlight EADA as an interesting alternative to TTC in the context of tranplantation markets. We conclude by focusing on a key policy goal in kidney-exchange markets: the number of transplantations.

We focus on special kidney-expchange problems. We assume that participants are all incompatible with their directed donor (see, e.g., Sönmez et al., 2018), and therefore a transplantation only occurs if they are assigned a kidney that they rank higher than that of their directed donor (i.e., their private endowment). We then compare the number transplantations resulting from TTC and EADA fixing the policymaker's priorities. The results are ambiguous, and thus trivially generalize to the more general case of where the exchange also manages a queue (Roth et al., 2004).[18]

**Proposition 7.** *There exist special kidney-exchange problems in which DA and EADA produce strictly more (fewer) transplantations than TTC.*

*Proof.* To see that DA and EADA can generate more transplantations than TTC, consider the following kidney-exchange problem. There are three agents and three directed donor kidneys: objects with capacity 1, each agent owns a single object. Let $(P, \succ)$ as follows:

| $P_1$ | $P_2$ | $P_3$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $b$ | $a$ | $a$ | 1 | 2 | 3 |
| $c$ | $c$ | $b$ | 3 | 3 | 1 |
| $a$ | $b$ | $c$ | 2 | 1 | 2 |
| 1 | 2 | 3 | | | |

[18] A similar result for the case of pure public endowment follows from Afacan and Dur (forthcoming).

TTC selects $\mu^1$—a two-way exchange generating 2 transplantations. DA and EADA both select $\mu^2$—a three-way exchange leading to 3 transplantations.

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^1$ | $b$ | $a$ | $c$ |
| $\mu^2$ | $b$ | $c$ | $a$ |

For the other direction, consider a kidney-exchange problem with four agents. Let $(P, \succ)$ as follows:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|---|---|---|---|
| $b$ | $a$ | $b$ | $b$ | 1 | 2 | 3 | 4 |
| $a$ | $c$ | $d$ | $c$ | $\vdots$ | 4 | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | 3 | | |
| | | | | | 1 | | |

TTC selects $\nu^1$ generating 4 transplantations. EADA selects $\nu^2$ generating 3 transplantations only. Since EADA Pareto dominates DA, DA generates no more than 3 transplantations as well.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\nu^1$ | $b$ | $a$ | $d$ | $c$ |
| $\nu^2$ | $a$ | $c$ | $d$ | $b$ |

$\square$

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# Appendix A Omitted Proofs

## A.1 Proof of Theorem 2

Here, we show that the EADA matching is t-justified-envy minimal in the set of Pareto-efficient matchings. For completeness, Appendix B contains a nearly identical proof for p-justified-envy minimality.

**Lemma 2.** *Fix an allocation problem. If $\mu$ and $\mu'$ are Pareto efficient, and $\mu'$ has less t-justified envy than $\mu$, then there exists $(i, j, \mu(j)) \in JE^{\mu}$ such that $\mu'(i) = \mu(j)$.*

*Proof.* Toward contradiction, suppose this is not the case, i.e., if $(i, j, \mu(j)) \in JE^{\mu}$, then $\mu'(i) \neq \mu(j)$. Since $\mu'$ has less t-justified envy than $\mu$, there exists t-justified envy $(i, j, \mu(j))$ such that $(i, j, \mu(j)) \in JE^{\mu}$ and $(i, j, \mu(j)) \notin JE^{\mu'}$. Thus, one of the following cases must occur.

**Case 1: $\boldsymbol{\mu'(i) P_i \mu(i)}$.** First, Pareto efficiency of $\mu$ implies $\mu'(i) \neq i$, and $\mu'(i)$ fills its capacity at $\mu$. Hence, since $\mu'$ assigns $i$ to $\mu'(i)$, there must exist agent $k_1 \in I$ such that $\mu(k_1) = \mu'(i)$ and $\mu'(k_1) \neq \mu'(i)$. Second, note that $\mu'(i)$ must be filled with agents with higher priority than $i$ at $\mu$. Otherwise, there is an agent $k$ such that $(i, k, \mu'(i)) \in JE^{\mu}$, in contradiction to the assumption that $\mu'$ has less t-justified envy than $\mu$. So, $k_1 \succ_{\mu'(i)} i$. Third, since $JE^{\mu'} \subsetneq JE^{\mu}$ and $(k_1, i, \mu'(i)) \notin JE^{\mu}$, $(k_1, i, \mu'(i)) \notin JE^{\mu'}$. Hence, $\mu'(k_1) P_{k_1} \mu'(i)$. By the same argument, there exists agent $k_2$ such that $\mu(k_2) = \mu'(k_1)$, $k_2 \succ_{\mu'(k_1)} k_1$, and $\mu'(k_2) P_{k_2} \mu(k_1)$. Iterating the argument, we get a sequence of agents that strictly prefer each others' allocation under $\mu$. Since $I$ is finite, this sequence must contain a cycle, contradicting Pareto efficiency of $\mu$.

**Case 2: $\boldsymbol{\mu(i) R_i \mu'(i)}$ and $\boldsymbol{\mu'(j) P_j \mu(j)}$.** This case is identical to Case 1 except for $i$ being replaced with $j$ in the argument.

**Case 3: $\boldsymbol{\mu(i) R_i \mu'(i)}$ and $\boldsymbol{\mu(j) P_j \mu'(j)}$.** By Pareto efficiency of $\mu'$, $\mu(j)$ must be full at $\mu'$. So, there exists $k_1 \in I \setminus \{j\}$ such that $\mu(k_1) \neq \mu(j)$ and $\mu'(k_1) = \mu(j)$. Since there does not exist new t-justified envy $(i, k_1, \mu(j))$ at $\mu'$, we must have $k_1 \succ_{\mu(j)} i$ and thus $k_1 \succ_{\mu(j)} j$. Moreover, if $\mu(j) P_{k_1} \mu(k_1)$, then $(k_1, j, \mu(j)) \in JE^{\mu}$, so we cannot assign $k_1$ to $\mu(j)$ at $\mu'$ since it contradicts the assumption on $\mu'$. We must have $\mu(k_1) P_{k_1} \mu(j)$. Note that $\mu(k_1)$ must be full at $\mu'$ since $\mu'$ is Pareto efficient. So, there exists new agent $k_2 \in I \setminus \{j, k_1\}$ such that $\mu(k_2) P_{k_2} \mu(k_1)$ and $\mu'(k_2) = \mu(k_1)$. Iterating this argument leads to a cycle that contradicts the Pareto efficiency of $\mu'$. □

We now turn to proving Theorem 2. Throughout the proof, let $R$ be the last round of the SEADA algorithm and $\alpha^r$ be a matching resulting from $r$-th round of the SEADA algorithm for $r = 0, 1, 2, \ldots, R$.

Towards contradiction, assume that there exists an allocation problem such that the EADA matching is not t-justified-envy minimal among Pareto-efficient matchings. Then there is another Pareto-efficient matching, $\beta$, has less t-justified envy than the EADA matching, $\alpha^R$. By Lemma 2, there exists $(i, j, \alpha^R(j)) \in JE^{\alpha^R}$ such that $\beta(i) = \alpha^R(j)$.

Let $r \in \{0, 1, 2, \ldots, R\}$ be the round of SEADA in which $i$ is removed. First, we claim that there exists $k_1 \in I \setminus \{i\}$ such that $\alpha^r(k_1) = \beta(i)$ and $\beta(k_1) P_{k_1} \beta(i)$. Since $i$ is rejected from $\beta(i)$ in the $r$-th round DA algorithm, $\beta(i)$ is full in the $r$-th round DA algorithm. If all agents whose assignment under $\alpha^r$ is $\beta(i)$ stay in the same object at $\beta$, there is no copy for $i$. Therefore, there must exist an agent $k_1 \in I \setminus \{i\}$ such that $\alpha^r(k_1) = \beta(i)$ and $\beta(k_1) \neq \beta(i)$. Note that $k_1 \succ_{\beta(i)} i$, and $(k_1, i, \beta(i)) \notin JE^{\alpha^R}$ since Lemma 2 of Tang and Yu (2014) guarantees that $\alpha^R(k_1) R_{k_1} \alpha^r(k_1) = \beta(i)$. Since there is no new t-justified envy at $\beta$, we have $\beta(k_1) P_{k_1} \beta(i)$.

Now, we claim that there exists $k_2 \in I \setminus \{i, k_1\}$ such that $\alpha^r(k_2) = \beta(k_1)$ and $\beta(k_2) P_{k_2} \beta(k_1)$. By the same reasoning as above, we can conclude that there exists $k_2 \in I \setminus \{k_1\}$ such that $\alpha^r(k_2) = \beta(k_1)$ and $\beta(k_2) P_{k_2} \beta(k_1)$. Now we want to show that $k_2 \neq i$. Note that $i$ was assigned to the underdemanded object in the $r$-th round DA algorithm, and $k_2$ was not. Hence, they cannot be the same.

Next, we claim that there exists $k_3 \in I \setminus \{i, k_1, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. By the same reasoning as above, we can conclude that there exists $k_3 \in I \setminus \{i, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. Now we want to show that we can find $k_3 \neq k_1$. Suppose that $\beta(k_2) = \alpha^r(k_1)$. Then we are additionally assigning two agents, $i$ and $k_2$, to $\alpha^r(k_1)$ at $\beta$. Therefore, there must be another agent $k_3$ other than $k_1$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. Therefore, finally, we have that there exists $k_3 \in I \setminus \{i, k_1, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$.

To have $\beta(k_3) P_{k_3} \alpha^r(k_3)$, there must exist a new agent $k_4 \in I \setminus \{i, k_1, k_2, k_3\}$ with the same conditions. The argument for the existence of $k_4$ is nearly identical to $k_3$. By iterating the argument, we can construct a sequence of distinct agents $\{k_1, k_2, k_3, \ldots\}$ such that $\beta(k_l) = \alpha^r(k_{l+1}) P_{k_l} \alpha^r(k_l)$ for $l = 1, 2, 3, \ldots$. Since $|I| < \infty$, this sequence must contain a cycle, contradicting Pareto efficiency of $\alpha^R$. $\square$

## A.2 Proof of Proposition 3

*Proof of Proposition 3.* The proof resembles the proof of Proposition 1 in Kesten (2010) except that we consider special kidney-exchange problems. If a mechanism is t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$, then it must select a Pareto-efficient and fair matching whenever one exists. Therefore, it suffices to show that a Pareto-efficient and strategyproof mechanism cannot satisfy this property even if we restrict allocation problems to special kidney-exchange problems.

Consider the following example. There are three agents and three objects with the capacity of 1. Let $(P, \succ)$ be as follows:

| $P_1$ | $P_2$ | $P_3$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $c$ | 3 | 2 | 1 |
| $b$ | $b$ | $a$ | 1 | 3 | 2 |
| $c$ | $c$ | $b$ | 2 | 1 | 3 |
| 1 | 2 | 3 | | | |

The priorities $\succ$ are consistent with each agent owning a single object (agent 1 owns $c$, agent 2 owns $b$, and agent 3 owns $a$). Fix $(I, S, q, o, \succ)$ throughout the proof. Suppose that a mechanism $\varphi$ is t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$, and strategyproof. For this allocation problem, $\varphi$ must select the matching $\mu^1$, which is the unique Pareto-efficient and fair matching (it can be computed using DA).

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^1$ | $a$ | $b$ | $c$ |

Suppose that agent 2 reports $P'_2 : a - c - b - 2$. By efficiency agent 2 must be assigned by $\varphi$, and by strategyproofness she is not assigned to $a$. If $\varphi$ assigns her to $c$ under the profile $(P_1, P'_2, P_3)$, then by Pareto efficiency, agents 1 and 3 are assigned to $a$ and $b$ respectively. Then, however, in the problem $(P_1, P'_2, P_3)$, agent 3 can benefit by misreporting her preferences as $P'_3 : a - \cdots$ since $\varphi$ must select the unique Pareto-efficient and fair matching $\mu^2$ for $(P_1, P'_2, P'_3)$:

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^2$ | $b$ | $c$ | $a$ |

Suppose $\varphi$ assigns agent 2 to $b$ under $(P_1, P'_2, P_3)$. Then 2 can benefit by misreporting her preferences as $P''_2 : c - \cdots$ in the problem $(P_1, P'_2, P_3)$ since $\varphi$ must select the unique Pareto-efficient and fair matching $\mu^2$ for $(P_1, P''_2, P_3)$. $\square$

## A.3 Proofs from Section 6

*Proof of Proposition 4.* First, note that any mechanisms in the priority-adjusted TTC class is strategyproof. Thus, by Proposition 3, such mechanism cannot be t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$, even when restricting attention to kidney-exchange problems. Furthermore, since C&T, FC&T, AC&T, ETTC and TTC are equivalent in one-to-one allocation problems, they too are not t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$.

Next, consider the allocation problem where $I = \{1, 2, 3, 4, 5\}$, $S = \{a, b, c, d, e\}$, each object has capacity of 1 and is owned by a different agent, and $(P, \succ)$ is as follows:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ | $\succ_e$ |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $d$ | $a$ | $d$ | $e$ | 2 | 1 | 3 | 5 | 4 |
| $b$ | $b$ | $d$ | $b$ | $d$ | 3 | 4 | $\vdots$ | 2 | 5 |
| $\vdots$ | $a$ | $c$ | $e$ | $\vdots$ | 1 | 2 | | 3 | $\vdots$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | | 4 | |
| | | | | | | | | $\vdots$ | |

Both DA+TTC and CPDA+TTC select the matching $\mu^1$, for which $JE^{\mu^1} = \{(2, 4, d), (3, 1, a), (3, 4, d)\}$, and $\widehat{JE}^{\mu^1} = \{(2, d), (3, a), (3, d)\}$. However, $\mu^2$ is also Pareto efficient with $JE^{\mu^2} = \{(3, 1, a)\} \subsetneq JE^{\mu^1}$, and $\widehat{JE}^{\mu^2} = \{(3, a)\} \subsetneq \widehat{JE}^{\mu^1}$. Thus, both mechanisms are not t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\mu^1$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $\mu^2$ | $a$ | $d$ | $c$ | $b$ | $e$ |

Finally, consider the allocation problem where $I = \{1, 2, 3\}$, $S = \{a, b, c\}$, each object has capacity of 1 and owned by each different agent, and $(P, \succ)$ is as follows:

| $P_1$ | $P_2$ | $P_3$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | 1 | 3 | 2 |
| $b$ | $a$ | $b$ | 3 | 2 | $\vdots$ |
| $c$ | $c$ | $c$ | 2 | 1 | |
| 1 | 2 | 3 | | | |

B and AB select the matching $\mu^3$ with $JE^{\mu^3} = \{(3, 2, b)\}$, and $\widehat{JE}^{\mu^3} = \{(3, b)\}$. However, the matching $\mu^4$ is Pareto efficient and fair. Thus, both mechanisms are not t-justified-envy (p-justified-envy) minimal in $\Phi^{PE}$.

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^3$ | $a$ | $b$ | $c$ |
| $\mu^4$ | $a$ | $c$ | $b$ |

□

*Proof of Proposition 5.* To prove a similar result, the working paper version of Abdulkadiroğlu et al. (2020) constructs Pareto-efficient and strategyproof mechanisms that select the same matchings as priority-adjusted TTC, C&T, FC&T, and ETTC respectively except for some allocation problems. For these allocation problems, the mechanisms they constructed do not violate individual rationality, and they induce no justified envy while priority-adjusted TTC, C&T, FC&T, and ETTC do. Thus, these mechanisms are individually rational and they induce less p- and t-justified envy than priority-adjusted TTC, C&T, FC&T and ETTC respectively, which completes the proof. □

*Proof of Proposition 6.* Examples where these mechanisms create less justified-envy than TTC are given in Morrill (2015) and Hakimov and Kesten (2018). For the other direction, consider the allocation problem where $I = \{1, 2, 3, 4\}$, $S = \{a, b, c, d\}$, $q_a = q_c = q_d = 1$ and $q_b = 2$, objects are publicly owned, and $(P, \succ)$ is as follows:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|---|---|---|---|
| a | b | a | c | 2 | 1 | 3 | 1 |
| c | ⋮ | c | d | 4 | 2 | 1 | ⋮ |
| d | | ⋮ | ⋮ | 1 | ⋮ | 4 | |
| ⋮ | | | | 3 | | ⋮ | |

TTC yields the fair matching $\mu^1$. AC&T, C&T, FC&T, and ETTC all select matching $\mu^2$ with $JE^{\mu^2} = \{(1, 3, a), (1, 4, c)\}$, and $\widehat{JE}^{\mu^2} = \{(1, a), (1, c)\}$, completing the proof.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu^1$ | a | b | c | d |
| $\mu^2$ | d | b | a | c |

□

# Appendix B Additional materials

## B.1 Proof of Theorem 2 with respect to p-justified-envy Minimality[19]

**Lemma 3.** *Fix an allocation problem. If $\mu$ and $\mu'$ are Pareto efficient, and $\mu'$ has less p-justified envy than $\mu$, there exists $(i,s) \in \widehat{JE}^{\mu}$ such that $\mu'(i) = s$.*

*Proof.* Toward contradiction, suppose this is not the case, i.e., if $(i,s) \in \widehat{JE}^{\mu}$, then $\mu'(i) \neq s$. Since $\mu'$ has less p-justified envy than $\mu$, there exists p-justified envy $(i,s)$ such that $(i,s) \in \widehat{JE}^{\mu}$ and $(i,s) \notin \widehat{JE}^{\mu'}$. Thus, one of the following cases must occur.

**Case 1: $\boldsymbol{\mu'(i) P_i \mu(i)}$.** First, Pareto efficiency of $\mu$ implies $\mu'(i) \neq i$, and $\mu'(i)$ fills its capacity at $\mu$. Hence, since $\mu'$ assigns $i$ to $\mu'(i)$ at $\mu'$, there must exist agent $k_1 \in I$ such that $\mu(k_1) = \mu'(i)$ and $\mu'(k_1) \neq \mu'(i)$. Second, note that $\mu'(i)$ must be filled with agents with higher priority than $i$ at $\mu$. Otherwise, $(i, \mu'(i)) \in \widehat{JE}^{\mu}$, in contradiction to the assumption that $\mu'$ has less p-justified envy than $\mu$. So, $k_1 \succ_{\mu'(i)} i$. Third, since $\widehat{JE}^{\mu'} \subsetneq \widehat{JE}^{\mu}$ and $(k_1, \mu'(i)) \notin \widehat{JE}^{\mu}$, $(k_1, \mu'(i)) \notin \widehat{JE}^{\mu'}$. Hence, $\mu'(k_1) P_{k_1} \mu'(i)$. By the same argument, there exists agent $k_2$ such that $\mu(k_2) = \mu'(k_1)$, $k_2 \succ_{\mu'(k_1)} k_1$, and $\mu'(k_2) P_{k_2} \mu(k_1)$. Iterating the argument, we get a sequence of agents that strictly prefer each others' allocation under $\mu$. Since $I$ is finite, this sequence must contain a cycle, contradicting Pareto efficiency of $\mu$.

**Case 2: $\boldsymbol{\mu(i) R_i \mu'(i)}$ and $\boldsymbol{\mu'(j) P_j \mu(j)}$.** This case is identical to Case 1 except for $i$ being replaced with $j$ in the argument.

**Case 3: $\boldsymbol{\mu(i) R_i \mu'(i)}$ and $\boldsymbol{\mu(j) P_j \mu'(j)}$.** By Pareto efficiency of $\mu'$, $s$ must be full at $\mu'$. So, there exists agent $k_1 \in I \setminus \{j\}$ such that $\mu(k_1) \neq s$ and $\mu'(k_1) = s$. Since $(i,s) \notin \widehat{JE}^{\mu'}$, we must have $k_1 \succ_s i$ and thus $k_1 \succ_s j$. Moreover, if $s P_{k_1} \mu(k_1)$, then $(k_1, s) \in JE^{\mu}$, so we cannot assign $k_1$ to $s$ at $\mu'$ since it contradicts the assumption on $\mu'$. We must have $\mu(k_1) P_{k_1} s$. Note that $\mu(k_1)$ must be full at $\mu'$ since $\mu'$ is Pareto efficient. So, there exists new agent $k_2 \in I \setminus \{j, k_1\}$ such that

[19] For the case of p-justified envy, one can use the characterization results of Tang and Zhang (2020) for a shorter proof. Tang and Zhang (2020) show that a mechanism Pareto dominates any mechanism with fewer blocking pairs if and only if it coincides with the results of EADA (potentially with different, profile specific, sets of consenters). Since EADA is Pareto efficient only with full consent (the case described in this paper), and since efficient mechanisms are non-wasteful, this implies that EADA (with full consent) is p-justified-envy minimal in the class of Pareto-efficient mechanisms. This holds true as otherwise another Pareto-efficient mechanism would have less p-justified envy, but this would mean that it has fewer blocking pairs and so that it is Pareto dominated by EADA, a contradiction to the other mechanism's optimality.

$\mu(k_2) \neq \mu(k_1)$ and $\mu'(k_2) = \mu(k_1)$. Iterating this argument leads to a cycle that contradicts the Pareto efficiency of $\mu'$. $\square$

We now turn to proving that the EADA matching is p-justified-envy minimal among Pareto-efficient matchings. Throughout the proof, let $R$ be the last round of the SEADA algorithm and $\alpha^r$ be a matching resulting from $r$-th round of the SEADA algorithm where $r = 0, 1, 2, \ldots, R$.

Towards contradiction, assume that the EADA matching is not p-justified-envy minimal in the set of Pareto-efficient matchings. Then there is another Pareto-efficient matching, $\beta$, has less p-justified envy than the EADA matching, $\alpha^R$. By Lemma 3, there exists $(i, s) \in JE^{\alpha^R}$ such that $\beta(i) = s$.

Let $r \in \{0, 1, 2, \ldots, R\}$ be the round of SEADA in which $i$ is removed. First, we claim that there exists $k_1 \in I \setminus \{i\}$ such that $\alpha^r(k_1) = \beta(i)$ and $\beta(k_1) P_{k_1} \beta(i)$. Since $i$ is rejected from $\beta(i)$ in the $r$-th round DA algorithm, $\beta(i)$ is full in the $r$-th round DA algorithm. If all agents whose assignment under $\alpha^r$ is $\beta(i)$ stay in the same object at $\beta$, there is no copy for $i$. Therefore, there must exist an agent $k_1 \in I \setminus \{i\}$ such that $\alpha^r(k_1) = \beta(i)$ and $\beta(k_1) \neq \beta(i)$. Note that $k_1 \succ_{\beta(i)} i$, and $(k_1, \beta(i)) \notin \widehat{JE}^{\alpha^R}$ since Lemma 2 of Tang and Yu (2014) guarantees that $\alpha^R(k_1) R_{k_1} \alpha^r(k_1) = \beta(i)$. Since there is no new p-justified envy at $\beta$, we have $\beta(k_1) P_{k_1} \beta(i)$.

Now, we claim that there exists $k_2 \in I \setminus \{i, k_1\}$ such that $\alpha^r(k_2) = \beta(k_1)$ and $\beta(k_2) P_{k_2} \beta(k_1)$. By the same reasoning as above we can conclude that there exists $k_2 \in I \setminus \{k_1\}$ such that $\alpha^r(k_2) = \beta(k_1)$ and $\beta(k_2) P_{k_2} \beta(k_1)$. Now we want to show that $k_2 \neq i$. Note that $i$ was assigned to the underdemanded object in the $r$-th round DA algorithm, and $k_2$ was not. Hence, they cannot be the same.

Next, we claim that there exists $k_3 \in I \setminus \{i, k_1, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. By the same reasoning as above, we can conclude that there exists $k_3 \in I \setminus \{i, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. Now we want to show that we can find $k_3 \neq k_1$. Suppose that $\beta(k_2) = \alpha^r(k_1)$. Then we are additionally assigning two agents, $i$ and $k_2$, to $\alpha^r(k_1)$ at $\beta$. Therefore, there must be another agent $k_3$ other than $k_1$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$. Therefore, finally, we have that there exists $k_3 \in I \setminus \{i, k_1, k_2\}$ such that $\alpha^r(k_3) = \beta(k_2)$ and $\beta(k_3) P_{k_3} \beta(k_2)$.

To have $\beta(k_3) P_{k_3} \alpha^r(k_3)$, there must exist a new agent $k_4 \in I \setminus \{i, k_1, k_2, k_3\}$ with the same conditions. The argument for the existence of $k_4$ is nearly identical to the argument for the existence of $k_3$. And by iterating this argument, we can construct a sequence of distinct agents $\{k_1, k_2, k_3, \ldots\}$ such that $\beta(k_l) = \alpha^r(k_{l+1}) P_{k_l} \alpha^r(k_l)$ for $l = 1, 2, 3, \ldots$. Since $|I| < \infty$, this sequence must contain a cycle, contradicting Pareto efficiency of $\alpha^R$. □

## B.2 TTC Variants

**Clinch and Trade (Morrill, 2015)**

**Step 0** For each object $s$, assign a counter $c_s$ and initialize it to $c_s = q_s$.

**Step 1** **1.a** Agent $i$ is immediately assigned to object $s$ if her first-choice among options that were not removed is $s$, and she has the $c_s$ highest priority at $s$. We call this *clinching an object*. In this case, remove $i$ and set $c_s = c_s - 1$. If $c_s = 0$, then remove $s$. If $i$'s first choice among options that were not removed is herself, remove $i$ and leave her unassigned. We call this *self-clinching*. Repeat clinching and self-clinching procedures until no agent can.

**1.b** Remaining agents and objects point to their first-choices among options that were not removed. Every agent in a cycle (defined in the TTC algorithm) is removed with a copy of the object she points to. For each object $s$ in the cycle, set $c_s = c_s - 1$. If $c_s = 0$, remove $s$. Proceed to the next step.

In general, at

**Step k, k $\geq$ 1** **k.a** If agent $i$ pointed to object $s$ in Step $k-1$ and $s$ is not removed yet, let $i$ keep pointing to $s$. If $s$ is removed, $i$ joins clinching and self-clinching procedures. Only for these agents, repeat clinching and self-clinching procedures until no agent can.

**k.b** Remaining agents and objects point to their first-choices among options that were not removed. Every agent in a cycle is removed with a copy of the object she points to. For each object $s$ in the cycle, set $c_s = c_s - 1$. If $c_s = 0$, remove $s$. Proceed to the next step.

The algorithm terminates when all agents are removed from the allocation problem. We call the mechanism that associates to every allocation problem the outcome of the above algorithm C&T. Morrill (2015) shows that this mechanism is Pareto efficient and strategyproof.

**Always Clinch and Trade (Morrill, 2015)**

Note that the C&T algorithm allows agent $i$ to clinch object $s$ only before she points to it. If we allow agent $i$ to clinch to object $s$ whenever her first-choice among available objects is $s$ and she has the $c_s$ highest priority at $s$ among remaining agents, then it is called the Always Clinch and Trade (AC&T) algorithm. We call the mechanism that corresponds to the AC&T algorithm AC&T. Morrill (2015) showed that this mechanism is Pareto efficient but not strategyproof.

**First Clinch and Trade (Morrill, 2015)**

The last version Morrill (2015) introduced is the First Clinch and Trade (FC&T) algorithm. The main difference between the C&T algorithm and the FC&T algorithm is that we do not update the set of agents who can clinch objects in the FC&T algorithm. Recall that this set changes in the C&T algorithm as steps go on. The FC&T algorithm runs as follows:

**Step 0** For each object $s$, assign a counter $c_s$ and initialize it to $c_s = q_s$.

**Step k, k $\geq$ 1** If $c_s = 0$, remove object $s$. If agent $i$'s first-choice among options that were not removed is herself, remove $i$ and leave her unassigned. Remaining agents and objects point to their first-choice among remaining options. If $i$ is pointing at $s$ and she initially had the $q_s$ highest priority at $s$, then assign $i$ to $s$ immediately, remove $i$, and set $c_s = c_s - 1$. For the remaining agents and objects, every agent in a cycle is removed with a copy of the object she points to. For each object $s$ in the cycle, set $c_s = c_s - 1$. Proceed to the next step.

The algorithm terminates when all agents are removed from the allocation problem. We call the mechanism that corresponds to the above algorithm FC&T. Morrill (2015) shows that this mechanism is also Pareto efficient and strategyproof.

**Equitable TTC (Hakimov and Kesten, 2018)**

**Step 0** Create agent-object pairs by letting each object $s \in S$ assign its $q_s$ copies to agents based on its priority order. For agent $i \in I$, if her first-choice is herself, remove all of pairs including her and leave her unassigned. The copies paired with her remain to inherited. Each remaining agent-object pair $(i, s)$ points to the agent-object pair $(i', s')$ if (1) $s'$ is $i$'s first-choice among paired objects, and (2) $i'$ has the highest priority of $s$ among agents paired with $s'$.

If $i$ is paired with her first-choice object, then let all pairs including her point to that pair. There must exist at least one cycle (including self-cycles). For each cycle, remove all agent-object pairs in the cycle, assigning each agent to the object that is in the pair she points to. Note that an agent might appear at multiple pairs in the same or different cycles. In that case, assign $i$ to her best choice and the other copies of that object remain to be inherited. For each pair $(i, s)$ in a cycle, the copies paired with $i$ in other pairs that do not participate in a cycle remain to be inherited as well.

**Step k, k $\geq$ 1 Inheritance** If object $s$ has copies that remain to be inherited from Step $k-1$ and there are no existing pairs of $s$ from Step $k-1$, i.e., all pairs of $s$ are removed in Step $k-1$, then $s$ assigns its copies that remain to be inherited to remaining agents following its priority order and create new pairs.

**Pointing and trading** For agent $i$, if her first-choice is herself, remove all of pairs including her and leave her unassigned. The copies paired with her remain to inherited. Each remaining agent-object pair $(i, s)$ points to the agent-object pair $(i', s')$ if (1) $s'$ is $i$'s first-choice among paired objects, and (2) $i'$ has the highest priority of $s$ among agents paired with $s'$. If $i$ is paired with her first-choice object, then let all pairs including her point to that pair. There must exist at least one cycle (including self-cycles). For each cycle, remove all agent-object pairs in the cycle, assigning each agent to the object that is in the pair she points to. Note that an agent might appear at multiple pairs in the same or different cycles. In that case, assign $i$ to her best choice and the other copies of that object remain to be inherited. For each pair $(i, s)$ in a cycle, the copies paired with $i$ in other pairs that do not participate in a cycle remain to be inherited as well.

The algorithm terminates when all agents are removed. We call the mechanism that corresponds to the above algorithm ETTC. Hakimov and Kesten (2018) shows that this mechanism is also Pareto efficient and strategyproof.

## B.3 The Boston Mechanism and its Variant

### The Boston Mechanism (Abdulkadiroğlu and Sönmez, 2003)

**Step 1** Each agent $i$ proposes to the first-choice. If $i$'s first-choice is herself, then she is removed and left unassigned. Each object $s$ accepts up to $q_s$ applicants following its priority order, and rejects the rest.

In general, at

**Step k, k $\geq$ 2** Each agent $i$ who has not been accepted nor removed proposes to the $k$-th choice. If $i$'s $k$-th choice is herself, then she is removed and left unassigned. Each object $s$ accepts agents up to the number of remaining copies following its priority order, and rejects the rest.

The algorithm terminates when there are no more agents/objects, and agents are assigned to the objects that accepted them. We call the mechanism that corresponds to the above algorithm the Boston (B) mechanism.

**The Adaptive Boston Mechanism (Mennle and Seuken, 2014)**

**Step 1** Each agent $i$ proposes to the first-choice. If $i$'s first-choice is herself, then she is removed and left unassigned. Each object $s$ accepts up to $q_s$ applicants following its priority order, and rejects the rest.

In general, at

**Step k, k $\geq$ 2** Each agent $i$ who has not been accepted nor removed proposes to the most preferred choice with available slots (including proposing to herself). If she proposes to herself, then she is removed and left unassigned. Each object $s$ accepts agents up to the number of remaining slots following its priority order, and rejects the rest.

The algorithm terminates when there are no more agents/objects, and agents are assigned to the objects that accepted them. We call the mechanism that corresponds to the above algorithm the Adaptive Boston (AB) mechanism.

## B.4 Independence of Essential Stability (Troyan et al., 2020)

Troyan et al. (2020) provide another justification for EADA. They define *the reassignment chain initiated by p-justified envy* $(i, s)$. In this chain, if agent $i$ claims a copy of object $s$, then an agent with the lowest priority at object $s$ becomes unassigned. And then she claims the best object among objects where she has justified envy. If repeating this process cycles back to the agent who initiated the chain, they say that the mechanism is *essentially stable*. Theorem 1 of Troyan et al. (2020) states that EADA is essentially stable.

While the argument of Troyan et al. (2020) may appear to be similar to our proof of Theorem 2, in the sense that assigning $i$ to $s$ yields undesirable consequences in both cases, the following two examples show that a Pareto-efficient matching can be t-justified-envy and p-justified-envy minimal in $\mathcal{M}^{PE}$ but not essentially stable, and that the opposite is also possible.

**Example** (A Pareto-efficient matching which is t-justified-envy and p-justified-envy minimal in $\mathcal{M}^{PE}$ but not essentially stable)**.** *Recall the example from the proof of Proposition 2 and $\mu^3$ for this allocation problem.*

| $P_1$ | $P_2$ | $P_3$ | $\succ_a$ | $\succ_b$ | $\succ_c$ |
|---|---|---|---|---|---|
| $b$ | $a$ | $a$ | 1 | 3 | 2 |
| $a$ | $c$ | $b$ | 2 | 1 | 3 |
| $c$ | $b$ | $c$ | 3 | 2 | 1 |
| 1 | 2 | 3 | | | |

*We know that $\mu^3$ is t-justified-envy and p-justified-envy minimal in $\mathcal{M}^{PE}$.*

| | 1 | 2 | 3 |
|---|---|---|---|
| $\mu^3$ | b | a | c |

*However, it is not essentially stable. If agent 3 claims object $b$, and then agent 1 claims object $a$, then agent 2 claims object $c$. Hence, the chain does not come back to 3.*

**Example** (A Pareto-efficient matching which is essentially stable but neither t-justified-envy nor p-justified-envy minimal in $\mathcal{M}^{PE}$)**.** *Suppose that $I = \{1, 2, 3, 4\}$ and $S = \{a, b, c, d\}$. Each object has capacity of 1. $(P, \succ)$ is as follows:*

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|---|---|---|---|
| $b$ | $b$ | $c$ | $a$ | 1 | 3 | 2 | 4 |
| $a$ | $d$ | $b$ | $d$ | 4 | 1 | 3 | 2 |
| $\vdots$ | $c$ | $\vdots$ | $\vdots$ | $\vdots$ | 2 | $\vdots$ | $\vdots$ |
| | $\vdots$ | | | | $\vdots$ | | |

*We can assume any ownership structure that is consistent with $\succ$. Note that matching $\mu^1$ is Pareto efficient and essentially stable.*

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu^1$ | $a$ | $b$ | $c$ | $d$ |

*The only t-justified envy is* $(1, 2, b)$ *and the only p-justified envy is* $(1, b)$*. If agent* 1 *claims object* $b$*, then agent* 2 *claims object* $c$*. Then agent* 3 *claims* $b$*, so the chain cycles back to agent* 1*. However, matching* $\mu^2$ *is Pareto efficient and has no t-justified envy (p-justified envy).*

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu^2$ | $b$ | $d$ | $c$ | $a$ |

*Hence,* $\mu^1$ *is not t-justified-envy (p-justified-envy) minimal in* $\mathcal{M}^{PE}$ *for this allocation problem.*