Stability vs. No Justified Envy

Assaf Romm Alvin E. Roth Ran I. Shorrer*

October 4, 2021

Abstract

Stability and "no justified envy" are used almost synonymously in the matching theory literature. However, they are conceptually different and have logically separate properties. We generalize the definition of justified envy to environments with arbitrary school preferences, feasibility constraints, and contracts, and show that stable allocations may admit justified envy. When choice functions are substitutable, the outcome of the deferred acceptance algorithm is both stable and admits no justified envy.

*First draft: January 2020. Romm: Department of Economics and Federmann Center for the Study of Rationality, The Hebrew University of Jerusalem, and Department of Economics, Stanford University, *email*: assaf.romm@mail.huji.ac.il. Roth: Department of Economics, Stanford University, *email*: alroth@stanford.edu. Shorrer: Department of Economics, Penn State University, *email*: rshorrer@gmail.com. This paper benefited from discussions with Fuhito Kojima, Scott Duke Kominers, Déborah Marciano, and Yannai Gonczarowski. The work of Assaf Romm is supported by the Maurice Falk Institute, by Israel Science Foundation (ISF) grant 1780/16, and by Koret Young Israeli Scholars Program. Romm and Shorrer are supported by United States–Israel Binational Science Foundation (BSF) grant 2016015.

1 Introduction

Stability and "no justified envy" are used almost synonymously in the matching theory literature. However, they have different logical meanings, and they reflect different concerns of market designers. Stability is defined as the property that no coalition can profitably deviate from a matching (Gale and Shapley, 1962). This definition is motivated by the concern that following a centralized matching process, some agents may deviate together, thus hindering the implementation of the intended outcome. Stability is widely considered a key determinant of the success or failure of a centralized clearinghouse (Roth, 1990).

The concept of no justified envy was introduced by Abdulkadiroğlu and Sönmez (2003) and is akin to a fairness condition proposed by Balinski and Sönmez (1999). Justified envy arises when a single agent is convinced she is prioritized over another agent, and prefers the outcome of that other agent to her own. Consider, for example, a public-school seat-allocation scenario. Student i is prioritized over student j at school s (e.g., because she lives closer to the school, and schools prioritize students according to proximity). If j is assigned to s, while i is assigned to a school she likes less, i experiences justified envy. In this case, it may be unlikely that student i will deviate together with public school s, but it is not unlikely that she will file an appeal or even argue her case in court.[1]

Despite these different motivations, stability and no justified envy are sometimes used interchangeably.[2] In some situations they indeed go hand in hand. In the context of school choice, where schools all have responsive preferences—such as those described by a capacity and a ranking of students—Abdulkadiroğlu and Sönmez (2003) show that a stable allocation admits no justified envy. Conversely, an individually rational and non-wasteful matching that admits no justified envy is

[1] For a recent case ruling on a similar issue regarding the school-choice system in Amsterdam, see: `http://uitspraken.rechtspraak.nl/inziendocument?id=ECLI:NL:RBAMS:2015:4085&` (in Dutch, retrieved February 3, 2020).

[2] As an example, consider Abdulkadiroğlu et al. (2020) and Dogan and Ehlers (forthcoming) who study the same setting and describe the same property under different labels (justified-envy minimal and minimal instability, respectively).

stable. However, the two concepts differ in more general domains. In the context of college admissions, for example, colleges often target balancing the composition of the incoming cohort across many dimensions such as diversity, academic focus, etc. Although in the last two decades the matching market design literature dedicated much attention to constrains and to the presence of contracts, the relation between stability and justified envy in these environments remains largely unexplored.

An example of how feasibility constraints draw a wedge between stability and no justified envy comes from no-spouse and anti-nepotism employment policies. These policies prevent family members from being hired by or admitted to the same institution. While illegal in most states of the United States today (as they may be considered a form of marital status discrimination), these kind of rules were prevalent in the past in the private and the public sector,[3] and they are still common in other countries.[4] When such employment policies are in place, an institution may consider two family members as the two most desirable candidates, but end up hiring only one of them, together with another, less desirable candidate. In this case, one can argue whether or not the family member not hired experiences justified envy toward the less qualified candidate who was hired.[5]

A good example for why it is important to distinguish between stability and no justified envy arises from Hungarian college admissions. Biró et al. (2020) study this matching-with-contracts environment, where students are ordered based on academic achievement. The current mechanism assigns students to schools *and* allocates funding based on the same ordering. This makes merit a criterion along which students may compare their outcomes with others,[6] and based on which students may experience justified envy toward their peers. Biró et al. (2020) show

[3]For a survey of case studies and a historical overview, see Chandler et al. (2002).

[4]In Israel, for example, anti-nepotism employment regulations are relatively common and exist in all branches of public service (Koch Davidovich, 2010).

[5]In the context of multi-dimensional constraints, Delacrétaz et al. (2019) note that a weak form of no justified envy is independent of stability.

[6]In this case, our definition of priority is indeed the same as the academic merit ordering (with the exception of top-performers who cannot be compared, see Example 3.2).

that the mechanism is stable and admits no justified envy, but also that there are other stable allocations that do admit justified envy and may be considered superior according to some criteria. For example, the number of students admitted to college could be increased by 2% by selecting another stable allocation. This provides some indication that policymakers—in Hungary and in other countries with similar circumstances—care not only about stability, but also about other properties, among them no justified envy.

Formally defining justified envy in more general environments (with arbitrary preferences, feasibility constraints, and contracts) is not straightforward. In the examples above the merit order is complete and exogenously given. But it is not always clear who is prioritized over whom, and as a consequence what type of envy is "justified." For example, in the context of business school admissions, schools are often classifying applicants into a few "types" (finance, consulting, entrepreneurs, etc.). While the choice between applicants of different types may depend on the composition of the rest of the cohort, it may be possible to point, for example, to the best candidate among the finance candidates. For a slightly more elaborate scenario (one that involves contracts), consider a college that uses one ranking for admissions and another ranking to determine how to allocate funding. It is not necessarily clear that the first student in line for funding is prioritized (in any meaningful sense) over any of the other students. She may be the last student in the admissions ranking. Moreover, both rankings may not be observable to an outside analyst.

Our approach is to first infer a priority relation from schools' observable choices in the spirit of the revealed preferences literature.[7] Schools' choices (represented by their choice functions) are determined by their preferences and potentially also by feasibility constraints imposed by a regulator. Choice functions are therefore a general way to describe both incentives and policy goals. We derive a priority relation that coincides with schools' rankings in the "standard" case of responsive preferences (but in general it is not a total order over students). Second, we

[7]Conceptually related is the literature on revealed preference in matching markets (See Chambers and Echenique, 2016, Chapter 10 and references therein).

generalize the definition of justified envy to cases where "justification" is given by a priority relation, like the one we infer. We then study the connection between stability and no justified envy in a model of matching with contracts (as in Hatfield and Milgrom, 2005) and constraints (see, e.g., Kamada and Kojima, forthcoming, and the references therein).

In this general model, the implication from stability to no justified envy depends on the properties of the schools' choice functions and on the strength of the derived priority relation. For example, if for all schools the priority relation is empty (i.e., no student is prioritized over any other student), there can be no justified envy.

When schools only have one type of position, no stable allocation admits justified envy. However, feasibility constraints may call for a different, more permissive priority relation. We consider a weak priority relation based on hypothetical school preferences that a student can imagine and that cannot be falsified by the school's observed choices (e.g., "I was only rejected from the school because the government imposed an affirmative action policy"). A stable allocation may admit weak justified envy, i.e., based on this weaker priority relation.

In the presence of multiple contract terms, a stable allocation may also admit justified envy. In some markets, this is true for *all* stable outcomes. However, we show that when schools' choice functions are substitutable, this cannot happen. Substitutability is a well-known property of choice functions, roughly stating that removing a contract from the choice menu does not make another contract less desirable. Following Hatfield and Milgrom (2005), it has been used extensively in market design to guarantee the existence of a stable allocation in many-to-one matching markets with contracts. When choice functions are substitutable, a stable allocation that admits no justified envy always exists. One such allocation can be reached using the student-proposing deferred acceptance (DA) algorithm (Gale and Shapley, 1962), providing yet another justification for using this mechanism.

The paper is structured as follows. Section 2 presents the model. Section 3 introduces our derivation of the revealed priority relation and its weaker counterpart, and presents results for generalized school-choice settings (without contracts). Section 4 covers the matching-with-contracts environment. Section 5 concludes.

2 Model

We begin by introducing a few definitions that will help us describe many-to-one matching markets with arbitrary preferences, feasibility constraints, and contracts.

Let I be a non-empty finite set of students, S a non-empty finite set of schools,[8] and T a non-empty finite set representing potential contract terms between students and schools. We denote the set of contracts by $X = I \times S \times T$. For a contract $x = (i, s, t) \in X$, we denote by x_I the student that participates in this contract, i, and by x_S the school that participates in this contract, s. For any subset of contracts $Y \subseteq X$, we denote the subset of contracts in which a given student $i \in I$ appears by $Y_i := \{x \in Y \mid x_I = i\}$, and the subset of contracts in which a given school $s \in S$ appears by $Y_s := \{x \in Y \mid x_S = s\}$. Whenever Y_i is a singleton and there is no risk of confusion, we also refer to the single contract as Y_i.

Each student i has a strict preference order $\succ_i$ over X_i and being unmatched (denoted by $\emptyset$). When explicitly writing a student's preference, we sometimes omit contracts less preferred to $\emptyset$. Each school s has a strict preference order $\succ_s$ over subsets of X_s (including the empty set). We assume throughout that schools never rank a subset that contains two or more contracts with the same student above the empty set.[9] For a school s, we denote by $\mathcal{P}_s$ the domain of strict preference orders over subsets of X_s that conform to this restriction.

Each school is also subject to an exogenously given constraint that takes the form of a collection of sets of contracts $\mathcal{F}_s \subseteq 2^{X_s}$. We say that a subset $Y \subseteq X_s$ is

[8] Although our results are motivated by applications with complex preferences, such as college admissions, we choose to use the traditional label "schools."

[9] Our discussion is adapted to the many-to-one matching model, rather than to the many-to-many matching environment.

feasible for s if $Y \in \mathcal{F}_s$, and that it is infeasible otherwise. We assume that the empty set is feasible, $\emptyset \in \mathcal{F}_s$. We denote by Ch_s the choice function induced by $\succ_s$ and[10,11] $\mathcal{F}_s$:

$$\forall X' \subseteq X, \mathrm{Ch}_s(X') := \max_{\succ_s} \{Y \subseteq X'_s \mid Y \in \mathcal{F}_s\}.$$

We use Ch_s^* to refer to the choice function of school s in the absence of feasibility constraints (i.e., with $\mathcal{F}_s$ replaced by 2^{X_s} in the expression above).

We refer to the tuple $\left(I, S, T, \{\succ_i\}_{i \in I}, \{\succ_s\}_{s \in S}, \{\mathcal{F}_s\}_{s \in S}\right)$ as a **market**. We now define a few well-known properties of choice functions.

Definition 2.1. The choice function Ch_s is **responsive** if there exists $q_s \in \mathbb{N}$ and a ranking of the "acceptable" contracts in X_s: $x_1 >_s x_2 >_s \cdots >_s x_n$, such that for any $X' \subseteq X$ the chosen contracts are the q_s maximal (acceptable) contracts in X' according to $>_s$, and multiple contracts with the same student are not chosen twice. If fewer than q_s such contracts are available, all of them are chosen. We say that school s has **responsive preferences** if Ch_s^* is responsive, i.e., in the absence of feasibility constraints, its preferences induce a responsive choice function.

Definition 2.2 (Hatfield and Milgrom, 2005)**.** Let R_s be the rejection function defined by $R_s(X') = X' \setminus \mathrm{Ch}_s(X')$. The choice function Ch_s is **substitutable** if for all subsets $X' \subseteq X'' \subseteq X$ we have $R_s(X') \subseteq R_s(X'')$.

An **allocation** is a collection of contracts, $Y \subseteq X$, where no student appears in multiple contracts, i.e., $\forall i \in I, |Y_i| \leq 1$. An allocation is feasible if $Y_s \in \mathcal{F}_s$ for each $s \in S$.

We say that student i blocks allocation Y if $\emptyset \succ_i Y_i$. Similarly, school s blocks allocation Y if $\mathrm{Ch}_s(Y_s) \succ_s Y_s$. We say that a coalition consisting of students I'

[10] An alternative approach would be to take choice functions as the primitives of our model. While perfectly useful for most of this paper, this approach is not suitable for our discussion on justified envy under feasibility constraints in Section 3.3.

[11] As schools' choice functions are derived from strict preferences and feasibility constraints, they automatically satisfy the irrelevance of rejected contracts condition (Aygün and Sönmez, 2013).

and school s blocks allocation Y if there exists a non-empty subset of contracts $X' \subseteq I' \times \{s\} \times T$ such that $\forall i \in I', |X'_i| = 1$, $X'_i \succ_i Y_i$, and $X' \subseteq \mathrm{Ch}_s(Y_s \cup X')$. Allocation Y is **stable** if it is not blocked by any student, school, or coalition.

In general, defining stability in the presence of constraints is not straightforward. A key difficulty is that some blocking pairs (or coalitions) may lead to infeasible allocations. We address this issue by integrating the constraints into the choice function, practically ruling out this kind of blocking coalitions. Similar solutions are suggested by Kamada and Kojima (2017) for markets with distributional constraints, and by Ehlers et al. (2014) for school choice with upper and lower quotas.

3 Generalized School Choice

In this section we only consider the case of $|T| = 1$ (the "generalized school-choice settings"). We omit any reference to the (single) term involved in the contracts. Furthermore, for any school s we abuse notation and treat $\succ_s$ as if it was an order over subsets of students, and treat Ch_s as if its domain and range were subsets of students. We return to the general setting in Section 4.

3.1 The Priority Relation

We define an extension of the priority relation used when discussing school choice with responsive preferences. We derive this priority relation from schools' observed behavior (as reflected by their choice functions), and intentionally avoid directly using the preferences of schools, as these may not be observable.

Definition 3.1. For any two students $i, j \in I$ and school $s \in S$, we say that i is **prioritized** over j at s, and write $i \gg_s j$, if

(a) $\forall I' \subseteq I$, $j \in \mathrm{Ch}_s(I' \cup \{i, j)\}) \Rightarrow i \in \mathrm{Ch}_s(I' \cup \{i, j\})$, and

(b) $\exists I' \subseteq I$ such that $i \in \mathrm{Ch}_s(I' \cup \{i, j\})$ and $j \notin \mathrm{Ch}_s(I' \cup \{i, j\})$.

In words, student i is prioritized over student j if whenever the school chooses j while i is available, the school chooses i as well, and the opposite does not hold (i.e., there exists a case where both students are available but only i is chosen). The priority relation (at a certain school) is an asymmetric binary relation over students. This relation is transitive if Ch_s is substitutable, but otherwise it may be non-transitive (as we prove in Appendix A).[12] Example 3.2 compares the priority relation we defined with the ranking of students under responsive preferences.

Example 3.2 (Responsive preferences)**.** Suppose that school s has responsive preferences with a capacity q_s and ranking $i_1 > \cdots > i_n$, and $\mathcal{F}_s = 2^I$. Then for any $q_s < t_2 \leq n$ and any $t_1 < t_2$, we have that $i_{t_1} \gg_s i_{t_2}$. Also, any acceptable student i_t is prioritized over any unacceptable student $j \in I \setminus \{i_1, \ldots, i_n\}$. This corresponds to our natural interpretation of the school's ranking as a priority relation, with the caveat that the school's choice data does not reveal any information about comparisons between the top q_s students (who are all guaranteed a place in the school).[13]

The notion of revealed priority may be useful for purposes other than defining justified envy. For example, the reserve design literature studies situations where

[12] One may be tempted to consider "conditional priority" along the following lines. Imagine a school with one slot reserved for minorities and a ranked list of students, some of whom belong to the minority population. According to our definition, while the top minority student is prioritized over all other students who are ranked below the number of available places, the second from the top minority student is already not prioritized over any of the students ranked above her, nor are any of them prioritized over her. However, conditional on the top minority student not being available, the second from the top minority student seems to have a justified claim over the slot reserved for minorities. We note that such motivation is more removed from the notions of "priority" or "justification" and closer to the notion of stability, since it takes into account coalitional deviations from a proposed allocation.

[13] Incidentally, using our priority relation simplifies the definition of Ergin's (2002) acyclity condition. A market contains a cycle if there exist $i, j, k \in I$ and $a, b \in S$ such that $i \gg_a j \gg_a k \gg_b i$. Ergin's main result is that when schools' choice functions are responsive, a fair and efficient allocation is guaranteed to exist regardless of student preferences if and only if the market contains no cycle.

a certain number of seats are reserved for specific groups of applicants (e.g., underrepresented minorities), while other seats are available to the general population. Dur et al. (2018) show that choice functions that try to first place students in general population seats benefit the prioritized group relative to choice functions that try to first place students in reserved seats. Consistently, students from the prioritized group (general population) are revealed preferred to more (fewer) students in this case. Similarly, the recent reform in order in which H1-B visas applications are processed (see Pathak et al., 2020) has led to more applicants with graduate degrees being prioritized over applicants who do not possess such degrees, even though the exogenous priority ranking was not changed. The examples of reservation policies highlight the fact that even when an order of merit is available, schools' choices may be inconsistent with this order, and lower merit individuals may be revealed preferred to higher merit individuals.

3.2 Justified Envy

We say that student i has **justified envy** toward student j at allocation Y if $[Y_j]_S \succ_i [Y_i]_S$, that is, i prefers the school to which j is assigned over her own assignment, and she is prioritized over j at that school. We say that Y **admits no justified envy** if there are no students i and j such that i has justified envy toward j at Y.

Unlike other definitions of justified envy that appear in the literature (e.g., Sönmez, 2013), our definition does not make use of an exogenously given priority relation, and instead uses the *revealed* priority. This makes our approach more flexible and appropriate for a variety of environments and preferences. The example of Educational Option programs in the New York City High School Match (Abdulkadiroğlu et al., 2005) provides a stark illustration of the difference between the two approaches. Educational Option programs are allowed to individually choose students for half of their seats, subject to the restriction that 16 percent be allocated to top performers in a standardized English Language Arts (ELA) exam, 68 percent to middle performers, and 16 percent to lower performers. Suppose a program s ranks students solely based on their ELA score, and chooses subsets of students subject to the above constraint. If student i has a better ELA score than student j, but student j has the highest score among "low performers," a possible outcome is that j gets admitted to s while i is not. In this case, a definition based on students' scores will deem i's envy towards j justified (reflecting a point of view that higher score students have higher priority for school seats). By contrast, by our definition j is prioritized over i at s, since j is always chosen by s when she is available, and therefore i's envy is not justified (reflecting the school's admission policy which guarantees a seat for j but not for i).

It is worthwhile to mention that extending the concept of no justified envy to arbitrary preferences is already enough to set it apart from stability, as is demonstrated by Example 3.3.

Example 3.3. Let $I = \{i_1, i_2, i_3, i_4\}$, and $S = \{s\}$. School s has two seats, and its preferences are given by

$$\{i_1, i_3\} \succ_s \{i_2, i_4\} \succ_s \{i_3, i_4\} \succ_s \{i_1, i_4\} \succ_s \{i_2, i_3\} \succ_s$$
$$\{i_1, i_2\} \succ_s \{i_1\} \succ_s \{i_2\} \succ_s \{i_3\} \succ_s \{i_4\} \succ_s \emptyset.$$

There are no feasibility constraints ($\mathcal{F}_s = 2^I$). Here $\text{Ch}_s(\{i_1, i_3, i_4\}) = \{i_1, i_3\}$, so i_4 is not prioritized over i_3. Furthermore, whenever i_2 and i_4 are both available, either both or none of them are chosen. This means that i_4 is also not prioritized over i_2.

Suppose students' preferences are such that i_1 finds s unacceptable, but i_2, i_3, and i_4 find s acceptable. Then the allocation $\{(i_2, s), (i_3, s)\}$ is individually rational,[14] non-wasteful,[15] and admits no justified envy, but it is not stable. This stands in contrast to the characterization result that holds for responsive preferences.

[14]We say that allocation Y is individually rational if for every student i, $Y_i \succ_i \emptyset$, and for every school s, $\text{Ch}_s(Y_s) = Y_s$.

[15]We say that student i claims an empty seat at school s under allocation Y if $s \succ_i Y_i$ and $\text{Ch}_s(Y_s \cup \{i\}) = Y_s \cup \{i\}$. Allocation Y is non-wasteful if no student claims an empty seat at any school under Y.

Despite this negative result, we show that the other direction of Abdulkadiroğlu and Sönmez's (2003) observation about responsive preferences does continue to hold in generalized school-choice settings. The following result does not impose any structure on the schools' preferences.

Proposition 3.4. *In generalized school-choice settings, a stable allocation admits no justified envy.*

Proof. Let Y be a stable allocation. Suppose that $[Y_j]_S = s \succ_i [Y_i]_S$. By stability, $i \notin \mathrm{Ch}_s(Y_s \cup \{i\})$. By stability and the irrelevance of rejected contracts property, $j \in \mathrm{Ch}_s(Y_s) = \mathrm{Ch}_s(Y_s \cup \{i\})$. Thus i does not have priority over j at s. □

3.3 Feasibility Constraints

We use feasibility constraints to model any technical rules or regulations that prohibit schools from choosing certain subsets of students independently of schools' preferences. Apart from the anti-nepotism employment policies mentioned in Section 1, feasibility constraints may also arise from affirmative action policies that impose bounds on the number of chosen students from each population group (Abdulkadiroğlu, 2005). In the presence of feasibility constraints, one should arguably use a slightly more nuanced definition of justified envy. The issue is that a student (who is aware of the feasibility constraints) may justifiably feel envy toward another student if the former would have been prioritized over the latter *in the absence of feasibility constraints.* If the preferences of the schools are observable, it is clear who is prioritized over whom, and that a stable allocation may exhibit justified envy.[16] However, if only schools' choices and feasibility constraints are observed (but not their "unmasked" preferences), a student may experience justified envy to an even greater extent if she forms a theory (that is consistent with the school's observed behavior) under which she is prioritized over other students.

Our definition below of the weak priority relation is motivated by considering this last scenario, in which schools' true preferences are masked by the feasibility

[16] In the same way that the COSM mechanism with ROTC priorities is not fair (Sönmez, 2013).

constraints.[17] We allow students to consider a candidate school preference relation, under which they see themselves as prioritized over other students, as long as this does not contradict with the observed choices. Until the end of this section, we will explicitly write $\mathrm{Ch}_s[\succ_s, \mathcal{F}_s]$ when referring to a school's choice function with preference relation $\succ_s$ and feasibility constraints $\mathcal{F}_s$. Similarly, we will write $\mathrm{Ch}_s[\succ_s]$ to refer to the choice function of school s with preference relation $\succ_s$ and in the absence of feasibility constraints.

Definition 3.5. For any two students $i, j \in I$ and a school $s \in S$, we say that i is **weakly prioritized** over j at s, and write $i \gg_s^w j$, if there exists some $\succ' \in \mathcal{P}_s$ such that

(1) $\forall I' \subseteq I$, $\mathrm{Ch}_s[\succ_s, \mathcal{F}_s] = \mathrm{Ch}_s[\succ'_s, \mathcal{F}_s]$,

(2a) $\forall I' \subseteq I$, $j \in \mathrm{Ch}^*_s[\succ'_s](I' \cup \{i, j)\}) \Rightarrow i \in \mathrm{Ch}^*_s[\succ'_s](I' \cup \{i, j\})$, and

(2b) $\exists I' \subseteq I$ such that $i \in \mathrm{Ch}^*_s[\succ'_s](I' \cup \{i, j\})$ and $j \notin \mathrm{Ch}^*_s[\succ'_s](I' \cup \{i, j\})$.

In words, student i is weakly prioritized over student j if there exists a school preference relation which (1) is indistinguishable from the true preference relation of s when masked by the feasibility constraints, and (2) prioritizes i over j, in the sense of Definition 3.1, in the absence of feasibility constraints. We again stress that the preference $\succ'$ appearing in Definition 3.5 is an *hyopthesized* preference, and the weak priority relation should be interpreted as *perceived* priority rather than true priority (see also Examples 3.6 and 3.7 below).

Clearly, $i \gg_s j \Rightarrow i \gg_s^w j$, since one can consider $\succ'_s$ which is the same as $\succ_s$, except for the sets not allowed by $\mathcal{F}_s$ (which are deemed less preferred to $\emptyset$ under $\succ'_s$). This means that more pairs of students are comparable according to the weak priority relation. Example 3.6 shows that the weak priority relation can sometimes be strictly finer, i.e., compare strictly more pairs. Example 3.7 further shows that $\gg_s$ may fail to be asymmetric.

[17] A similar consideration (the feasibility of certain blocking coalitions) motivates Ehlers and Morrill (2020) to weaken the stability notion.

Example 3.6. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s\}$. School s has responsive preferences with a capacity of 2, and it ranks i_1 above i_2 above i_3 above i_4.

$$\begin{aligned}\{i_1, i_2\} \succ_s \{i_1, i_3\} \succ_s \{i_1, i_4\} \succ_s \{i_2, i_3\} \succ_s \{i_2, i_4\} \succ_s \{i_3, i_4\} \succ_s \\ \{i_1\} \succ_s \{i_2\} \succ_s \{i_3\} \succ_s \{i_4\} \succ_s \emptyset.\end{aligned}$$

The feasibility constraint $\mathcal{F}_s = 2^I \setminus \{\{i_1, i_2\}, \{i_1, i_2, i_3\}, \{i_1, i_2, i_4\}, I\}$ reflects the fact that i_1 and i_2 cannot be admitted simultaneously.

With these preferences and constraints, $i_2 \not\gg_s i_4$, since $i_4 \in \mathrm{Ch}_s(\{i_1, i_2, i_4\})$ but $i_2 \notin \mathrm{Ch}_s(\{i_1, i_2, i_4\})$. However, $i_2 \gg_s^w i_4$. To see this, consider $\succ_s$, the true preference relation of s, without any feasibility constraints.

Example 3.7. Consider the same scenario of Example 3.6 with a different feasiblity constraint.

$$\mathcal{F}_s = \{\{i_1, i_3\}, \{i_2, i_4\}, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}, \emptyset\}.$$

This constraint restricts the school to choose either even-numbered or odd-numbered students, but not a mix of these two.

We get that $i_1 \gg_s^w i_2$ (by considering responsive preferences with the students being ordered $i_1 >_s i_3 >_s i_2 >_s i_4$) and also that $i_2 \gg_s^w i_1$ (by considering responsive preferences with the students being ordered $i_2 >_s i_1 >_s i_3 >_s i_4$, and with $\{i_1, i_3\} \succ_s \{i_2, i_4\}$).

We say that student i has **weak justified envy** toward student j at allocation Y if i prefers the school j is assigned to, $[Y_j]_S$, to her own assignment, and she is weakly prioritized over j at that school. We say that Y **admits no weak justified envy** if there are no students i and j such that i has weak justified envy toward j at Y.

A stable allocation may admit weak justified envy. For example, if in the settings from Example 3.6 we let s be acceptable to all students except student i_3, then $\{(i_1, s), (i_4, s)\}$ is the unique stable allocation, and i_2 has weak justified envy toward i_4 at this allocation.

Examples 3.6 and 3.7 show that the weak priority relation behaves erratically even when restricting consideration to small and more familiar domains. Moreover, when taken together with Example 3.2, our interpretation of these examples is that in the presence of feasibility constraints, not disclosing schools' priorities can leave a large number of students frustrated, believing that they were mistreated. This can be seen as a complement to Sönmez's (2013) analysis of *observable* priorities under feasibility constraints.

4 Multiple Contract Terms

We now return to the general matching-with-contracts model. Before we can define priority and justified envy for this environment, we need to consider the issue of interpersonal comparisons of contracts. When defining contracts, Hatfield and Milgrom (2005) do not consider interpersonal comparisons (such as envy) and consequentially their framework allows for a very general interpretation of terms of employment. For example, Juliet's terms of employment at a specific firm could be that she gets a salary of $150k and is allowed to bring a guest, but only if his name is Romeo, to the annual office party. This works out just great for Juliet. However, this kind of contract is not necessarily envied by Desdemona, who cannot enjoy the office party with Romeo, fearing that they would both suffer the wrath of her jealous husband Othello. Since we are interested in interpersonal comparisons, we look at settings where terms are comparable (e.g., getting a certain scholarship, being admitted to a certain study track, bringing a companion to the annual office party, and so on).

With this caveat in mind, we extend the definition of priority by referring to specific terms of employment that are relevant for both i and j.

Definition 4.1. For any two students $i, j \in I$ and school $s \in S$, we say that i is **prioritized** over j at s, and write $i \gg_s j$, if

(a) $\forall t \in T,\ \forall X' \subseteq X,\ (j,s,t) \in \mathrm{Ch}_s\left(X' \cup \{(i,s,t),(j,s,t)\}\right) \Rightarrow \mathrm{Ch}_s\left(X' \cup \{(i,s,t),(j,s,t)\}\right) \cap X_i \neq \emptyset$, and

(b) $\exists t \in T, \exists X' \subseteq X$ such that $(i,s,t) \in \text{Ch}_s(X' \cup \{(i,s,t),(j,s,t)\})$ and $\text{Ch}_s(X' \cup \{(i,s,t),(j,s,t)\}) \cap X_j = \emptyset$.

In words, if j is chosen with certain terms and there is a contract available with i under the same terms, then i is also chosen (under these terms, or others).[18] This definition is motivated by the idea that a student who is prioritized by the school will always be chosen to fill a specific position before a student over whom she is prioritized, unless the former is chosen to fill a different position.[19] For example, when contracts represent different study programs at a certain college, then a situation in which j is admitted to the economics program, and i is an applicant to the same program, indicates that i should be admitted to *some* program at that college (but, if i applied to multiple programs, not necessarily to the economics program).

As before, we say that student i has **justified envy** toward student j at allocation Y if i prefers the school and terms to which j is assigned to her own assignment, and she is prioritized over j at that school. Formally, suppose $(j,s,t) \in Y$, then i has justified envy toward j at Y if $(i,s,t) \succ_i Y_i$ and $i \gg_s j$.

Example 4.2 (Stable allocation with justified envy under substitutable choice functions)**.** Let $I = \{i,j,k\}$ and $S = \{s\}$. $T = \{\text{f}, \text{nf}\}$ represents funded and non-funded positions in the school. Denote $x_{i'}^t \equiv (i',s,t)$. The school has responsive preferences with capacity 2, and it prefers not to give funding, but cares lexicographically more about students' identities. In particular, it uses the ranking

$$x_i^{\text{nf}} > x_i^{\text{f}} > x_j^{\text{nf}} > x_j^{\text{f}} > x_k^{\text{nf}} > x_k^{\text{f}}.$$

[18] This definition is related to Sönmez's (2013) definition of fair branch priorities induced by a choice function. Sönmez asks whether a choice function violates a given priority order, whereas we first derive a priority from a given choice function.

[19] While Definition 4.1 is agnostic with regard to the interpretation of terms, an alternative definition of priority that assumes that terms are ordered will result in a (weakly) coarser priority relation. Such a relation could be more adequate for studying environments in which terms are naturally ordered, such as college admissions with funding (Hassidim et al., 2021) or cadet-branch matching (Sönmez, 2013; Sönmez and Switzer, 2013; Jagadeesan, 2019).

There are no feasibility constraints: $\mathcal{F}_s = X_s$. The induced choice function is substitutable, and $i \gg_s k$.

Suppose that student i's preferences are given by $x_i^{\mathrm{f}} \succ_i x_i^{\mathrm{nf}} \succ_i \emptyset$, student j finds all contracts unacceptable, and student k's preferences are $x_k^{\mathrm{f}} \succ_k \emptyset$. There are two stable allocations: $Y_1 = \left\{x_i^{\mathrm{f}}, x_k^{\mathrm{f}}\right\}$, which is the student-optimal stable allocation and which admits no justified envy, and $Y_2 = \left\{x_i^{\mathrm{nf}}, x_k^{\mathrm{f}}\right\}$, which is the school-optimal stable allocation and at which i has justified envy toward k.

While Example 4.2 may seem a bit contrived, it does resemble many existing college admissions environments, in which funding levels may vary between students, and not necessarily be based on their priority for the purpose of admission. The example provides simpler college preferences than are often used in practice, since in most college admissions scenarios funding is limited, which results in a non-substitutable choice function (see, e.g., Hassidim et al., 2017).

Theorem 4.3 below shows that for substitutable choice functions, while some stable allocations may induce justified envy, there always exists at least one stable allocation that does not admit justified envy. Furthermore, such a stable allocation can be found using the student-proposing DA.[20]

Theorem 4.3. *If all schools' choice functions are substitutable, then there exists a stable allocation that admits no justified envy, and one such allocation can be reached using the student-proposing DA.*

Proof. We consider the cumulative offer process, which, in the domain of substitutable choice functions, produces the student-optimal stable allocation (Hatfield and Milgrom, 2005). In the cumulative offer process, students propose as in DA, but schools get to choose from all the offers they received in the current or any previous round.

Let Y be the stable allocation that results from the cumulative offer process. Suppose that some student i has justified envy toward another student j at the allocation Y, and that $Y_j = (j, s, t)$.

[20] Example 4.2 illustrates that this result does not extend to the allocation resulting from school-proposing DA.

Since i envies j, and Y is stable, it must be that $(i, s, t) \notin \text{Ch}_s (Y \cup \{(i, s, t)\})$. But, since i has priority over j at s, it must be that $\text{Ch}_s (Y \cup \{(i, s, t)\}) \cap X_i \neq \emptyset$. Therefore there exists $t' \neq t$ such that $(i, s, t') \in \text{Ch}_s (Y \cup \{(i, s, t)\})$, and this implies that $(i, s, t') \in Y$, meaning i is assigned to s under different terms than j.

Since i is eventually assigned the contract (i, s, t') and she envies j, it must be that during the cumulative offer process i proposed to s the contract (i, s, t) and it was rejected. Let Y'_s denote the set of contracts that were available to s during the run of the cumulative offer process when (i, s, t) was first rejected. Note that Y'_s may contain contracts with i that were already rejected (and, by substitutability, are also rejected from Y'_s), and it may also contain contracts with j (at most one of which is not rejected). However, it cannot contain (j, s, t). The reason is that (j, s, t) is not rejected from a set that includes Y'_s (the set of offers available when the process terminates). Thus, by substitutability, had Y'_S contained (j, s, t), this contract would have been chosen from Y'_S, in contradiction to $i \gg_s j$.

Now consider the set of all contracts available to s at the end of the cumulative offer process, and denote it by Y''_s. We know that $(j, s, t) \in Y_s = \text{Ch}_s(Y''_s)$. Since $Y'_s \subseteq Y''_s$, substitutability implies that $(j, s, t) \in \text{Ch}_s (Y'_s \cup \{(j, s, t)\})$. Now $i \gg_s j$ implies that $\text{Ch}_s (Y'_s \cup \{(j, s, t)\}) \cap X_i \neq \emptyset$, which means that i was assigned one of the contracts she proposed. And, by substitutability, we get that the same contract should have been chosen from Y'_s as well, a contradiction. □

While Theorem 4.3 equally applies in the presence of feasibility constraints and in their absence, it should be noted that the theorem's applicability may be limited by such constraints in many cases. That is, given preferences that would induce a substitutable choice function absent any feasibility constraints, imposing the constraints on it may result in a non-substitutable choice function. This is, for example, the case with budget constraints (Mongell and Roth, 1986) and with capacity constraints (Romm, 2014, Lemma 1).[21]

Finally, we observe that outside of the substitutable preferences domain, The-

[21] See also Kojima et al. (2020) for a discussion on the effects of constraints on substitutability in a matching-with-salaries environment.

orem 4.3 fails to hold. Appendix A provides an example of a market with a single school whose choice function satisfies the law of aggregate demand (Hatfield and Milgrom, 2005), bilateral substitutability (Hatfield and Kojima, 2010), and substitutable completability (Hatfield and Kominers, 2015), where the *unique* stable allocation induces justified envy. This observation has important implications, as the two latter generalizations of substitutability are used extensively in studies of matching market design and in practice. The matter of whether or not a result similar to Theorem 4.3 holds for choice functions that satisfy unilateral substitutability (Hatfield and Kojima, 2010) turns out to be more elusive, and it is left open.[22]

5 Discussion

The elimination of justified envy is an important concept that embodies basic fairness considerations and that deserves the attention of market designers. Abdulkadiroğlu and Sönmez (2003) observed that it is tightly related to the absence of blocking pairs in the Boston Public Schools' new matching system, and showed that stability implies no justified envy in the context of school choice. This contribution inspired a large literature that employs this connection to study stable mechanisms and their fairness properties.

As demonstrated in this paper, when constraints or contracts are present, the two concepts of stability and no justified envy become independent. While we do not take a stand regarding the relative desirability of these properties, we note that this independence opens the door for designing market mechanisms that satisfy only one of these two properties. For example, as mentioned above, Biró et al. (2020) show that policymakers in Hungary choose a fair and stable mechanism, even when choosing a stable but unfair allocation could could increase the number of admitted students. A similar choice is made in many other countries, including Australia, Azerbaijan, and Turkey.

[22] Biró et al. (2020, Proposition 5) present a similar result for certain college admissions scenarios, where schools' choice functions are non-substitutable.

When choice functions are substitutable, using the deferred acceptance algorithm does lead to a stable allocation that also admits no justified envy. This may explain why, in many practical situations, no wedge between stability and no justified envy has been observed. It remains an open problem to identify a bigger family of choice functions that ensures the existence of a stable allocation that admits no justified envy. Similarly, the problem of finding an extensive family of preferences and feasibility constraints that induce substitutable choice functions also remains open.

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A Additional Proofs and Examples

A.1 Transitivity of the Priority Relation

We first look at the case of substitutable choice functions, and then provide an example for non-transitivity with a non-substitutable choice function.

Proposition A.1. *If* Ch_s *is substitutable, then* $\gg_s$ *is transitive.*

Proof. Suppose that $i \gg_s j$ and $j \gg_s k$; we need to show that $i \gg_s k$. For the first requirement, let $I' \subseteq I$ be such that $k \in \mathrm{Ch}_s(I' \cup \{i,k\})$. If $j \in I'$ then from $j \gg_s k$ we get that $j \in \mathrm{Ch}_s(I' \cup \{i,k\})$, and from $i \gg_s j$ we get that $i \in \mathrm{Ch}_s(I' \cup \{i,k\})$. Otherwise, let $I'' = I' \cup \{j\}$. If $k \notin \mathrm{Ch}_s(I'' \cup \{i,k\})$; then from the irrelevance of rejected contracts property it follows that $j \in \mathrm{Ch}_s(I'' \cup \{i,k\})$, from $i \gg_s j$ that $i \in \mathrm{Ch}_s(I'' \cup \{i,k\})$, and from substitutability that $i \in \mathrm{Ch}_s(I' \cup \{i,k\})$. If $k \in \mathrm{Ch}_s(I'' \cup \{i,k\})$, then from $j \gg_s k$, we get $j \in \mathrm{Ch}_s(I'' \cup \{i,k\})$, from $i \gg_s j$, we get $i \in \mathrm{Ch}_s(I'' \cup \{i,k\})$, and from substitutability, we get $i \in \mathrm{Ch}_s(I' \cup \{i,k\})$.

For the second requirement, let $I' \subseteq I$ be such that $i \in \mathrm{Ch}_s(I' \cup \{i,j\})$ and $j \notin \mathrm{Ch}_s(I' \cup \{i,j\})$. Consider $I'' = I' \cup \{k\}$. From substitutability, $j \notin \mathrm{Ch}_s(I'' \cup \{i,j\})$, and so from $j \gg_s k$ we also know that $k \notin \mathrm{Ch}_s(I'' \cup \{i,j\})$. The irrelevance of rejected contracts property then ensures that $i \in \text{Ch}_s(I'' \cup \{i,j\}) = \text{Ch}_s(I' \cup \{i,j\})$, and we are done. □

The following example shows that when choice functions are not substitutable, the priority relation may fail to be transitive.

Example A.2. Let $I = \{i,j,k\}$ and $S = \{s\}$. The preferences of s are given by

$$\{i,j,k\} \succ_s \{i,k\} \succ_s \{i\} \succ_s \{j\} \succ_s \{k\},$$

and there are no feasibility constraints ($\mathcal{F}_s = 2^I$). Here $i \gg_s j$ and $j \gg_s k$, but $i \not\gg_s k$.

A.2 Unique Stable Allocation that Admits Justified Envy under Non-Substitutable Preferences

Example A.3. Let $I = \{i,j,k\}$, $S = \{s\}$, and $T = \{A,B,C\}$. To ease notation, we will represent the contract (i,s,t) by i_t, and similarly for all other students. The school's preferences are given by

$$\begin{aligned}\{j_A,i_B,k_B\} &\succ_s \{j_A,i_C,k_B\} \succ_s \{j_A,k_B,i_A\} \succ_s \{j_A,i_B\} \succ_s \{j_A,i_C\} \succ_s \\ &\{j_A,k_B\} \succ_s \{j_A,i_A\} \succ_s \{j_A\} \succ_s \{i_B,k_B\} \succ_s \{i_B,j_B\} \succ_s \{i_C,k_B\} \succ_s \\ &\{i_C,j_B\} \succ_s \{k_B,j_B\} \succ_s \{k_B,i_A\} \succ_s \{j_B,i_A\} \succ_s \{i_B\} \succ_s \{i_C\} \succ_s \\ &\{k_B\} \succ_s \{j_B\} \succ_s \{i_A\} \succ_s \emptyset.\end{aligned}$$

These preferences satisfy the law of aggregate demand (Hatfield and Milgrom, 2005), bilateral substitutability (Hatfield and Kojima, 2010), and substitutable completability (Hatfield and Kominers, 2015). Substitutable completability and bilateral substitutability are (independent) weaker conditions than substitutability, but they still ensure the existence of a stable allocation. Both conditions are also weaker than Hatfield and Kojima's (2010) unilateral substitutability (Kadam, 2017).

To verify that Ch_s satisfies substitutable completability, one can consider a completion of $\succ_s$ that can be thought of as follows: the school always chooses j_A

when it is available, and in addition chooses from the remaining contracts according to a "responsive" choice function (in quotation marks as it allows choosing two contracts with the same student) with a capacity of 2 and the order over singletons $i_B > i_C > k_B > j_B > i_A$, and all other contracts (j_C, k_A, and k_C) are unacceptable. Formally we define $\bar{\succ}_s$ as (underlined allocations violate the constraint that each student can only be assigned one contract):

$$\underline{\{j_A, i_B, i_C\}} \bar{\succ}_s \{j_A, i_B, k_B\} \bar{\succ}_s \underline{\{j_A, i_B, j_B\}} \bar{\succ}_s \underline{\{j_A, i_B, i_A\}} \bar{\succ}_s \{j_A, i_C, k_B\} \bar{\succ}_s$$
$$\underline{\{j_A, i_C, j_B\}} \bar{\succ}_s \underline{\{j_A, i_C, i_A\}} \bar{\succ}_s \underline{\{j_A, k_B, j_B\}} \bar{\succ}_s \{j_A, k_B, i_A\} \bar{\succ}_s \underline{\{j_A, j_B, i_A\}} \bar{\succ}_s \{j_A, i_B\} \bar{\succ}_s$$
$$\{j_A, i_C\} \bar{\succ}_s \{j_A, k_B\} \bar{\succ}_s \underline{\{j_A, j_B\}} \bar{\succ}_s \{j_A, i_A\} \bar{\succ}_s \{j_A\} \bar{\succ}_s \underline{\{i_B, i_C\}} \bar{\succ}_s \{i_B, k_B\} \bar{\succ}_s$$
$$\{i_B, j_B\} \bar{\succ}_s \underline{\{i_B, i_A\}} \bar{\succ}_s \{i_C, k_B\} \bar{\succ}_s \{i_C, j_B\} \bar{\succ}_s \underline{\{i_C, i_A\}} \bar{\succ}_s \{k_B, j_B\} \bar{\succ}_s \{k_B, i_A\} \bar{\succ}_s$$
$$\{j_B, i_A\} \bar{\succ}_s \{i_B\} \bar{\succ}_s \{i_C\} \bar{\succ}_s \{k_B\} \bar{\succ}_s \{j_B\} \bar{\succ}_s \{i_A\} \bar{\succ}_s \emptyset.$$

Next, note that $i \gg_s j$: if j_A is chosen and i_A is also available, then either it or another contract with i is chosen. If j_B is chosen and i_B is also available, then i_B is also chosen. j_C is unacceptable, and so is never chosen. Finally, if we look at $X' = \{k_B\}$, then $i_B \in \mathrm{Ch}\left(X' \cup \{i_B, j_B\}\right)$, but $j_B \notin \mathrm{Ch}\left(X' \cup \{i_B, j_B\}\right)$.

Let the students' preferences be given by

$$i : i_A \succ_i i_C \succ_i \emptyset$$
$$j : j_B \succ_j j_A \succ_j \emptyset$$
$$k : k_B \succ_k \emptyset.$$

One can easily verify that the unique stable allocation is $\{i_C, j_A, k_B\}$, and under it i has justified envy toward j.